INDIAN SCIENTIFIC HERITAGE

DR. N. GOPALAKRISHNAN

FOREWORD BY DR. R.A. MASHELKAR, F.R.S.

Indian Institute of Scientific Heritage
Thiruvananthapuram

Heritage Publication Series -14

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About the author......

The author, Dr. N. Gopalakrishnan was born on 20th November, 1955. He took his M.Sc. Pharm. Chem (1978); M.Sc. Chemistry (1979); M.A. Industrial Sociology (1985) Degree in Journalism (1987) and Ph.D. in Biochemistry. He has been awarded the D. Lit (2002) for his outstanding contribution through the study of the scientific heritage of India. He is a Senior Scientist, in the Regional Research Laboratory, Council of Scientific and Industrial Research (CSIR) and Hon. Director of the Indian Institute of Scientific Heritage. He is the recipient of D.V. Memorial Award (1985), Gardners Award (1988), Dhingra Memorial Award and Gold medal (1990) and again D.V. Memorial Award (1993) for the achievements in the field of scientific research. He has been awarded the first NCSTC award for the popularisation of science, by the DST, Govt. of India, in 1988. He is also the recipient of the prestigious Canadian International Development Agency (CIDA) Fellowship of the Government of Canada in 1993 and was a visiting scientist in the University of Alberta, Edmonton, Canada and visited many universities in USA, Canada and many middle east countries for delivering lectures. He has six patents and fifty research papers in the scientific studies.

He has also received four awards for the literary works and has 41 books to his credit, both in scientific and cultural subjects and many popular articles on Indian Scientific Heritage.

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Indian Scientific Heritage

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Published by :

Indian Institute of Scientific Heritage (IISH)
Registered Charitable Trust 328/99/IV
Ushus, Estate Road, Pappanamcode
Trivandrum - 695 018 (Ph. 490149)

Rs. 250/- (US $ 25)

(For Members Rs. 125/-)

Printed at:

Sree Printers (DTP, Offset & Screenprinting)
Ind. Estate, Pappanamcode, TVM - 19, Ph. 490135

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Details of the bank account

Beneficiary : IISH Trivandrum
Ac No : 57020795171
IFSC : SBIN0070030
Bank : SBI industrial estate, papanamcode
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Thryaadyekothhara vruddhya yaathyeka: prathidinam narasthuanya: dasa yojanaani kiyathaa kaalena thayorgathisthulyaa

One man goes with an initial speed 3 yojanas per day and the rate of acceleration 1 yojana per day, and another man goes with the constant speed of 10 yojanas per day. In what time will they cover the same distance?

This problem is solved by the equation $n = 2(u - v)/f + 1$. Here 'n' is the number of days required for the meeting of the travellers. 'u' is the uniform speed of the second traveller and 'v' the initial speed of the first and 'f' is the acceleration of the first traveller per day. Among the moving bodies two problems are connected with the rate of their motion, i.e with acceleration and without acceleration and in the same direction and in the opposite direction of motion.

There is yet another type of motion in which the moving bodies make forward and backward motions at a fixed speed. If a body moves backward and forward at different speeds, then time taken to reach the destination is to be calculated using different speeds, then time taken to reach the destination is to be calculated using a different formula. Sreedharacharya in patiganita gives the rule (44.1)

On subtracting the backward motion (per day) from the forward motion per day, the true distance travelled per day will be obtained. (Dividing the distance by the true rate of distance travelled per day, one can find out the time). Bhaskaracharya I gives as exercise on this type of motion, in Bhaskarabhashya for Aryabhateeya (118.4)

नागोविंशति हस्तः प्रविशत्यर्धांगुलम् मुहूर्तेन ।
प्रत्येति च पञ्चांशं कतिभिरहोभिर्बिलं प्राप्तम् ।।

Naagovimsathihastha: pravisathyardhaangulam muhurthena
prathyethi cha panchaamsam kathi bhirahobhirbilam praaptham

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A serpent of 20 cubit long enters into the hole moving forward at the rate of ½ of an angula per muhurta and backward at the rate of 1/5 of an angula per muhurta. In how much time does the snake get into the hole completely?

Here the difference of forward and backward motions gives, the true forward distance travelled per unit time of muhurta. When total length of the serpent is divided by true forward distance, the time taken by the serpent to enter in the hole will be obtained.

A very interesting problem on the same topic is given from another old Sanskrit mathematical book. This is requoted in Patiganita book (example 32)

नागेन्द्रो दिनपञ्चमांशनवमत्र्यंशैः स्वपादान्वितैः
षड्भिर्याति सपादयोजनदलं त्र्यंशोनमर्धान्वितम् ।
प्रत्यायाति च योजनं द्विगुणितं स्वत्र्यंशहीनं सखे
सार्धाहेन च तत्र योजनशतं कालेन केनैष्यति ॥

Naagendro dina panchamamsanavamathryamsai: svapaadaanvithai:
shadbhiryaathi sapaadayojanadalam tryamsonamardhaanvitham
prathyaayathi cha yojanam dvigunitham svathryamsaheenam sakhe:
saardhaahena cha thathra yojanasatham kaalena kenaishyathi

The best amongst the elephants goes forward at a rate of (½ (1+1/4) (1-1/3) (1+½))of a yojana in 6x1/5x1/9x1/3(1+1/4) of a day and comes back at a rate of 2 (1-1/3) yojana in (1+½) days. In how much time will they go to a distance, 100 yojanas?

The rule applied here is the same as that of a body moving forward and backward. True distance travelled, foreward, per day is to be calculated from the difference and on dividing the distance by that value. The same problem appears in Ganita sara sangraha (V. 27)

Progression:

Progression is a chapter of great importance in

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mathematical calculations. Equations of both arithmetical and geometrical progression play important role in the calculation of many transactions and also estimation of areas and volumes of geometrical figures. This was a subject used for various applications in Indian mathematics too, even from the Vedic period. Arithmetic progression is given in Taitiriya samhita of the Yajurveda (VII. 2, 12 - 17)

एका चमे तिस्रश्चमे पञ्च चमे सप्त च मे नव च म एकादश
च मे त्रयोदश चमे पञ्चदश चमे

Ekaa chame thisraschame pancha chame saptha cha me
nava cha me ekaadasa cha me thrayodasa chame pancha dasa chame

It is the number order 1, 3, 5, 7, 9, 11, 13, 15, ... etc.

Another recension of the Yajurveda-Vajasaneyee Madhyandina samhita-also gives the arithmetic progression in even numbers. In another book namely Bruhadevata (500 BC), the result of the sum of arithmetic progression 1+2+3+4+5+.... + 100 is given as 5050. A mathematical progression is compared with an earthern drinking vessel with narrow bottom and steadily increasing upper opening. Patiganita (rule 79) says:

विस्तारोऽल्पोऽधस्तादुपरि महान् स्याद्यथा शरावस्य।
श्रेढीक्षेत्रस्य तथा गच्छसमो लम्बकस्तस्य ।।

Vistaaroalpodhastaaduperi mahaan syaadyathaa saraavasya
sreddeekshethrasya thathaa gacchasamo lambakasthasya

As in the case of an earthern drinking glass the base of which is smaller and the top wider, so also is the case with a series in progression figures. The altitude (lambaka) of that series figure is equal to the number of terms in the figure.

I.e. a progression generally starts with a small number or set and increases uniformly to large numbers.

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A variety of mathematical problems are given in ancient Indian books on the series/progressions. Many of the equations taught in modern mathematics, are given in these books. Patiganita (rule 14.1) gives an equation for calculating the sum of a basic arithmetic progression as follows:

सैकपदाहतपददलमेकादिचयेन भवति संकलितम् ॥

Saikapadaahathapadadalamekaadichayena bhavathi sankalitham

If the first term is unity and common difference is also unity, then the sum is equal to half the number of terms multiplied by total number of terms plus one.

I.e Sn of $1+2+3+\ldots+n = \frac{1}{2}n(n+1)$; where Sn is the sum and n is the number of terms.

A few (among a number of) problems, of different nature connected with determination of various parameters in the series are given to cite examples of the deep knowledge on progressions. Bhaskaracharya I in Aryabhateeyabhashya (105.1) has given this example:

आदिद्वितयं दृष्टं श्रेढ्याः प्रवदन्ति चोत्तरं त्रीणि ।
गच्छः पञ्च निरुक्तो मध्याशेषे धने ब्रूहि ॥

Aadidvithayam drushtam sreddyaa: pravadanthi chottharam threeni gaccha pancha niruktho madhyaasesbe dhane broohi

In a series, first term is 2; the common difference is stated to be 3 and number of terms is 5. Tell me the middle term and the sum of the series.

This example has also been given by Yallayya, Suryadeva and Raghunatha in their commentaries to Aryabhateeya (2.19). In Bhaskarabhashya another example (106.3) is given for finding out the desired term of a series in arithmetic progression.

एकादशोत्तरायाः सप्तादेः पञ्चविंशतिर्गच्छः।
तत्रान्त्योपान्त्यधने वद शीघ्रं विंशतेश्च कियत् ॥

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Ekaadasottharaayaa: saptaade: panchavimsathirgacche
thathraanthyopaanthya dhane vada seeghram vimsathescha kiyath

In an arithmetic progression, in which the common difference is 11, first term is 7, the number of terms is 25. Quickly say the ultimate and penultimate terms of that series and also say what is the 20th term.

Here the last term and 20th term are to be calculated using the equation: n^{th} term = a + (n-1)d. Where a is the first term and n is the total number of terms and d is the common difference.

When the first term and the last term of a series in an arithmetic progression are given, determining the number of terms is given in this exercise (Bhaskarabhashya example 107.6)

पञ्चभिराद्यः शङ्खः पञ्चोनशतेन यो भवेदन्त्यम्।
एकादश शंखानां यत्तन्मूल्यं त्वमाचक्ष्व ।।

Panchabhiraadya: sankha: panchonasathena yo bhavedanthyam
Ekaadasasankhaanaam yatthanmoolyam thvamaachakshva

Of 11 conch shells, which are arranged in increasing order of their prices (which are in arithmetic progression), the first shell is acquired for 5 and the last for 95. Say what is the price of the total shells.

Initial term (5), last term (95) and total terms (11) are indirectly given in the above problem, as prices and number of shells, respectively From this the common difference and sum are to be calculated using the equation given earlier.

Bhaskaracharya I also gives examples for finding out the number of terms when first term, sum and the common difference are given (Bhaskarabhashya 109.2)

नवकाष्टौ वृद्धिमुखे यत्र यत्कीर्त्यते धनं क्रमशः
रामाष्टशरैं दृष्टं पदप्रमाणं त्वया वाच्यम् ।।

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Navahaveetow vruddhimukha yathra yathheerthyathe dhanam kramena
namaashtaasaraa drushtam padapramaanam thwayaa vaachyam

In an arithmetic progression, common difference and the first term are 9 and 8 respectively, the sum is 583. Tell me the number of terms.

Using the first equation, answer can be found out, where Sn = 583, a = 8 and d = 9. This example is given for the application of the rule given in Aryabhateeya (2-19)

इष्टं व्येकं दलितं सपूर्वमुत्तरगुणं समुखमध्यम् ।
इष्टगुणितमिष्ट धनं त्वथवाद्यन्तं पदार्धहतम्

Ishtam vyekam dalitham sapoorvamuthaaragunam samukhamadhyam
ishtagunithamishta dhanam thwathavaadyantham padaardhahatham

Diminish the given number of terms by one, then divide by 2, then increase by the number of preceding terms (if any) then multiply by the common difference and increase by the first term of the series. Result is the arithmetic mean (of the given number of terms). This value multiplied by given number of terms is the sum of the given terms. Or, multiply the sum of the first and the last terms by half the number of terms i.e. Arithmetic mean of the series = a + (n-1)½ x d. Where a is the first term, n is the number of terms and d is the common difference.

Sum of the series = n (a + ½ (n-1)d) Or ½ n (a + s); s is the last term a and d are as explained above.

Another set of arithmetic progression which can be mathematically represented as follows is: a + (a + d) + (a + 2d) + ... (a + (n-1)d). The following rule is used for getting the sum of the progression series. (Patiganita 85 b)

श्रेढीक्षेत्रे तु फलं भूमुखयोगार्धलम्बहतिः ।

Sreddeekshethre thu phalam bhoomukhayogaardhalambahathi

The area of a progression (if it can be represented as a geometrical figure) is equal to the product of the half of the sum of the base and the face and the altitude. i.e Sn of $a+(a+d)+(a+2d)+\ldots\ldots(a+(n-1)d) = (a+\frac{1}{2}(n-1)d)n$

Then arithmetic mean of n terms is given as follows by converting the above form into another series, representing the mean (as per rule)

$$(a+pd)+(a+\underline{p+1}\times d)+\ldots\ldots+(a+\underline{p+(n-1)})d$$

Mean of the above series $= a + (\frac{1}{2}(n-1)+p)d$

When the above mean value is multiplied with the number of terms n, the sum of the series will be obtained, as follows. Sum of the series $= n\,(a+(\frac{1}{2}(n-1)+p)d)$. The same rule appears in Brahmasphuta siddhanta (XII. 42) and Ganitasara sangraha (VII.231.1a)

Aryabhatta I has given a complex equation for finding out the number of terms in a series, if the sum is known, in the above type of progression (Aryabhateeya 2-20)

गच्छोऽष्टोत्तरगुणिताद् द्विगुणाद्युत्तरविशेषवर्गयुतात् ।
मूलं द्विगुणाद्यूनं स्वोत्तरभजितं सरूपार्धम् ॥

Gacchashtottharagunithaad dvigunaadyutthara visesha vargayuthaath
moolam dvigunaadoonam svotthara bhajitham saroopaardham

Multiply the number of terms by 8 and by the common difference, increase that by the square of the difference between twice the first term and the common difference, and then take the square root, then subtract twice the first term, then divide by the common difference and add (to the quotient and take half the value, that gives the number of terms in a series of nature.

$$a+(a+d)+(a+2d)+(a+3d)+\ldots\ldots$$

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$$n = \frac{1}{2}a + \underline{(\sqrt{8ds} + (2a\text{-}d)^2 - 2a)} + d + 1$$

This rule has also appeared in Siddhanta sekhara (XIII. 24), Lilavati (rule 128) and Brahmasphuta siddhanta (XII. 18). In Patiganita (rule 87), Sreedharacharya gives an almost similar equation for calculating the number of terms.

अष्टोत्तरहतफलतो द्विगुणादिप्रचय विवरकृतियुक्तात् ।
मूलं द्विगुणमुखोनं सचयं द्विचयोद्धृतं गच्छः ॥

Ashtottharahataphalatho dviguṇaadi prachaya vivara kruthi yukthaath
moolam dvigunamukhonam sachayam dvichayoddhrutham gaccha:

Multiply the sum of the series by 8 times the common difference and to that product add the square of the difference between twice the first term and common difference. Take square root of that. That diminished by twice the first term and increased by the common difference and divided by twice the common difference gives the number of terms in the series i.e $n = (\sqrt{(8ds + (2a\text{-}d)^2} - 2a + d) + 2d$

Aryabhatta II in Mahasiddhanta (XV. 50) and Bhaskara II in Lilavati (128) have given another equation for the determination of the number of terms in a series:

$$n = (\sqrt{\underline{(2ds + (a\text{-}d/2)}^2} - a + d/2) \div d$$

Sripati in Siddhanta sekhara (XIII. 24) puts the above equation in this form also:

$$n = \underline{(\sqrt{(s/\tfrac{1}{2}\,d + (a\text{-}\tfrac{1}{2}d/d)^2}} - \underline{(a\text{-}\tfrac{1}{2}d)/d)}$$

In Patiganita (rule 14 ii) a method for finding the number of terms in an arithmetic progression when the initial term and the common difference are unity, is given.

द्विगुणीकृत सङ्कलितान्मूलं गच्छोऽवशिष्टसमम् ॥

Dviguneekrutha sankalithaanmoolam gacchovasishtasamam

The number of terms is equal to the square root of twice the sum of the series which must be the same as the residue left (after the extraction of square root). I.e $n = \sqrt{2Sn}$

For finding first term of the series Sreedharacharya has given a rule (Patiganita 86.I)

आदिः पदहृतगणितं निरेकगच्छघ्नचयदलेनोनम् ।।

Aadi:padahruthaganitham nirekagacchaghnachayadalenonam

The sum of a series, as divided by the number of terms of the series, being diminished by half the common difference as multipled by the number of terms minus 1, gives the first term of the series i.e First term $a = S/n - \frac{1}{2}d(n-1)$

In Patiganita the equation for finding the first term is given in rule 88

विपदपदवर्गदलाहतमिश्रघनात्फलमपास्य परिशिष्टं ।

व्येकपदार्धेन भजेद् व्येकेन पदाहतेनादिः ।।

Vipadapadavargadalaahathamisraghanaathphalamapaasya
parisishtam vyekapadaardhena bhajeth vyekena padaahathenaadi:

Having subtracted the sum of the series from the mixed amount as multiplied by one half of the number of terms squared minus the number of terms, divide the residue by one half of (the difference of) the number of terms minus one, as diminished by one and multiplied by the number of terms.

Thus the first term is $a = (\frac{1}{2}(n^2-1)(a+d)-S) + (\frac{1}{2}(n-1)-1)n$

Patiganita (86 ii) gives the rule for calculating the common difference of a series.

पदहृतफलं मुखोनं निरेकपददलहृतं प्रचयः ।

Padahruthaphalam mughonam nirekapadadalahrutham prachaya:

The sum of a series as divided by the number of terms (of

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the series) being diminished by the first term and then divided by half of the number of terms minus 1, gives common difference of the series.

I.e. Common difference $d = (S/(n-a)) + \frac{1}{2}(n-1)$. An applied problem is given in Bhaskarabhashyam (106.4)

द्व्यादित्र्युत्तर संख्यं दिने दिने कार्तिके क्रमान्मासे
प्रददाति महीपालः पञ्चदशाहे गतेविप्रः ।
ब्राह्मिष्ठः सम्प्राप्तस्तस्मै दत्ता दशाहधन संख्या
पञ्चदिनोत्थान्यस्मै कथय धनं किं तयोस्तत्र ॥

Dvyaadithriyuthara sankhyam dine dine kaarthike kramaanmaase
pradadaathi maheepaala: panchadasaahe gathervipra:
brahmishta: sampraapthasthasmai datthaa
dasaahadhana sankyaam panchadinotthaanyasmai
kathaya dhanam kim thayosthathra

In the month of karthika, a king gave away some money daily starting with 2 on the first day and increasing that by 3 per day. Fifteen days having passed, there arrived a Brahmana (Vedic scholar). The amount for the next 10 days was given to him and that for the next 5 days (of the month) to another. What do the last two person get?

Here the initial amount on the 15th day is to be calculated as the nth value in the arithmetic progression starting from 2 and incresing by 3. The total sum (S10-sum for 10 days- from 15th to 25 days are to be separately calculated as the first term the amount of 15th day and the common difference as 3 and the number of terms as 10. The same is to be repeated for the 26th to the 30th days (5 days) to calculate sum for 5 days (S5) for second person. Solution for getting the partial sums can thus be calculated.

Problem on the progression of $1 + (1+2)+(1+2+3)+$

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type of series. Bhaskaracharya has given many problems on this type of progression (109.1)

पञ्चानामष्टानां चतुर्दशानां च याः क्रमाच्चितयः ।
गच्छस्तराशित्रिकोणा (रूपविधानं च) मे वाच्यम् ॥

Panchaanaamashtaanaam chathurdasaanaam cha yaa: kramaacchitheya:
gacchastharaasthri konaa (roopavidhaanamcha) me vaachyam

There are (three pyramidal) piles (of balls) having respectively 5, 8 and 14 layers which are triangular. Tell me the number of units (balls) in each of them.

This problem is worked out as follows. In the topmost layer of pyramidal piles, there is 1 ball, in the second layer from the top, there are 1+2=3 balls; in the third layer 1+2+3=6; like wise in the 4th layer 10 balls and so on. Number of balls in the first pile having 5 layers, is equal to 1+(1+2)+....... upto fifth layer. The value is (5x6x7) 1/6. This formula is given in Aryabhateeya (2-21)

एकोत्तराद्युपचितेर्गच्छाद्येकोत्तरत्रिसंवर्गः ।
षड्भक्तः सचितिघनः सैकपदघनो विमूलो वा ॥

Ekottharaadyupachithergacchaadyekottharaathrisamvarga:
shadbhaktha: sachiti ghana: saikapada ghano vimoolo vaa

Of the series which has one for the first term and one for the common difference, take three terms in continuation, of which the first is equal to the given number of terms, and find their continues product. That product or the number of terms plus one subtracted from the cube of that divided by 6 gives the chitighana

Chitighana literally means, the solid content of a pile in the shape of a pyramid on a triangular base. The pyramid so constructed as 1 ball in the top and 1+2 balls next and so on. i.e.

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$n(n+1)(n+2)/6$ and $(n+1)^3 - (n+1)$ divided by 6 = Sn

Problem related with a series: $1^2+2^2+3^2+......n^2$ type. Aryabhateeya (2-22a) gives the rule for getting the sum of the squares of terms in a series.

सैकसगच्छपदानां क्रमात् त्रिसंवर्गितस्य षष्ठोंऽशः ।

Saikasagacchapadaanaam kramaath thri samvargithasya shashtomsa:

Continuous product of three quantities i.e the number of terms, plus one, the same increased by the number of terms and the number of terms when divided by 6 gives the sum of the series of squares of natural numbers (varga chiti ghana)

I.e for $1^2+2^2+3^2+.....n^2$ series. The sum = $n(n+1)(2n+1) \times 1/6$

The sum of the squares of the terms of the given series can be found out using the equation given by Sreedharacharya in Patiganita (rule 105)

$\Sigma (a+(r-1)d)^2 = (a+(a+2d)+a+4d)+....$to n terms) x a+ $(1^2+2^2+3^2+.... (n-1)^2$. This has been described as follows:

द्विगुणितचयेन गणितं मुखसङ्गुणितं निरेकगच्छस्य ।
कृतिसंकलितेन युतं चयकृतिगुणितेन वर्गयुतिः ।।

Dvigunitha chayena ganitham mukhasangunitham nirekagacchasya kruthisankalithena yutham chayakruthi gunithena vargayuthi:

The sum of the arithmetic series with twice the common difference when multiplied by the first term and then increased by the sum of the squares of natural numbers ranging from 1 to one less the number of terms, as multiplied by the square of the common difference, gives the sum of the squares of the terms of the given series.

Bhaskaracharya I gives an example for this type of series in

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Bhaskarabhashyam (111.1)

सप्तानां अष्टानां सप्तदशानां चतुर्भुजाश्चितयः ।
एकविद्यानां वाच्यं पदस्तरास्ता हि वर्गाख्याः ॥

Sapthaanaam ashtaanaam saptadasaanaam chathurbhujaaschithaya:
ekavidyaanaam vaachyam padastharaasthaa hi vargaakhyaa:

There are (three pyramidal) piles on square bases having 7, 8 and 17 layers which are also squares. Say the number of units there in.

There are three pyramids. In the topmost layer there is one brick, in the next layer there are four bricks (2^2), in the third layer 9, (3^2) bricks and so on Hence the number of bricks used in the three piles separetely are 140, 204 and 1785, respectively. The solution for this problem is possible from the equation given in Aryabhateeya (2-22) described above.

Problem of a series of the type: $1^3 + 2^3 + 3^3 + ... + n^3$.

The square of the sum of the series of natural numbers gives the sum of the series of cubes of natural numbers (Ghana chiti ghana) i.e. Sum of $1^3 + 2^3 + 3^3 + ... + n^3$ series $= (\frac{1}{2} n(n+1))^2$ $= (1+2+3+......+n)^2$

Bhaskaracharya I gives the problem in which the above formula is applied (111.2)

चतुरश्रघनश्चितयः पञ्चचतुर्नवस्तरा विनिर्देश्याः ।
एकावघटितास्ताः समचतुरश्रेष्टकाः क्रमशः ॥

Chathurasraghanaschithaya: panchachathurnavastharaa vinirdesyaa:
ekaavaghatithaasthaa: samachathura sreshtakaa: kramasa:

There are three pyramidal piles having 5, 4 and 9 cuboidal layers. They are cuboidal bricks (of unit dimension) with one brick in the topmost layer. Find the number of bricks used in them.

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There are 1^3 bricks on top and 2^3 in second layer 3^3 in the third layer and so on. The number of bricks in the three piles are 225, 100 and 2025, respectively. The Equation for solving the problem is given above. n, for three piles are respectively 5, 4 and 9. This rule is also given in the Patiganita (rule 103) for Σn^3.

सपदपदवर्गतोर्धं घनसंकलितं स्वसंगुणं भवति ।

Sapadapadavargathordham ghanasankalitham swasangunam bhavathi

One half of what is obtained by adding the number of terms to the square of the number of terms, when multiplied by itself, gives the sum of the cubes of natural numbers. (from 1 upto given number of terms) i.e $\Sigma n^3 = ((n^2+n)½)^2$

This rule is the same as that given by Aryabhatta and can be used to solve the problems like that given in Patiganita (117)

एकादिचयपदानां घनसंकलितं सखे कियद् भवति ।

अस्तु दशानां प्रकथय तथैव संकलित संकलितम् ॥

Ekaadichayapadaanaam ghanasankalitham sakhe kiyad bhavathi
aasu dasaanaam prakathaya thathaiva sankalitha sankalitham

Friend, quickly say what is the sum of cubes of 10 terms of a series whose 1st term and common difference are each unity. And sum of the successive sums of those terms.

Formula for finding out the value for the series of $\Sigma n + \Sigma n^2 + \Sigma n^3$ is given in Patiganita (rule 102 and 104)

द्विगुणितसैकपदघ्नं सैकपदं प(द)दलहृतं भवति ।

Dvigunitha saikapadaghnam saikapadam pa(da)dalahritham bhavathi

The number of terms plus one, as multiplied by twice the number of terms plus one, being (further) multiplied by half the number of terms.

$\Sigma n + n^2 + n^3 ... = ½\,(2n+1)(n+1)n$. The other rule is as follows.

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सैकपदवर्गाहितपदं द्विकोपेतगुणं भवति ।
संकलितकृतिघनानां संकलितैक्यं चतुर्भक्तं ॥

Saikapadavargathaadithapadam dvikopedagunam bhavathi
sankalithakruthighanaanaam sankalithaikyam chathurbhakrutham

The number of terms as multiplied by the square of (the sum of) the number of terms +1, when (further) multiplied by the number of terms plus 2 and divided by 4, gives (i) the sum of successive sums of natural numbers (from 1 upto the given numbers of terms) (ii) the sum of squares of those natural numbers and (iii) the sum of the cubes of those natural numbers. i.e $n(n+1)^2$ $(n+2)$ divided by 4.

Patiganita also gives another example as follows (118)

संकलितकृतिघनानां संकलितसमासमानां मे कथय।
षण्णां सखे पदानां गणयित्वा यदि विजानासि ॥

Sankalithakruthighanaanaam sankalithasamaasamaanaam me kathaya
shannaam sakhe padaanaam ganayithvaa yadivijaanaasi

Friend, if you know, then say after calculation (i) the sum of successive sum of 6 natural numbers (ii) the sum of the squares of the first 6 natural numbers and (iii) the sum of the cubes of first 6 natural numbers. (these can be calculated by above rules).

In Patiganita the sum of another series of the cubes of numbers is given (rule 107)

श्रेढीफलस्य वर्गे प्रचयहते (चय) विहीन वदनगुणम् ।
मुखफलवधं निदध्यादिष्टादिचयेन घनयोगः ॥

Sreddeephalasya varge prachayahathe (chaya) viheena vadanagunam
mukhaphalavadham nidadhyaadishtaadichayena ghanayoga:

To the square of the sum of the given arithmetic series, as multiplied by the common difference, and the product of the first and the sum of the series, as multiplied by the first term minus the common difference, the result is the sum of cubes of

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the terms of the series with given terms and common difference.

In a series $a^3 + (a+d)^3 + (a+2d)^3 + \ldots (a + (n-1)d)^3$,

The sum $= s^2 \times d + a \times s \times (a-d)$. Where $s = a + (n-1)d$

A problem on the application of this rule (Patiganita 121)

पञ्चादिद्विकवृद्धीनां पदानां ये क्रमात् घनाः ।
चतुर्णां तत्समासेन गणयित्वा निगद्यताम् ।।

Panchaadidvika vriddheenaam padaanaam ye kramaath ghanaa:
chathurnaam thathsamaasena ganayithva nigadyathaam

Say result of adding together the cubes of the four terms which begin with 5 and increase successively by 2.

Various types of rules / equations for calculating the sums, number of terms, common difference etc. in mathematical series, are given in many ancient books.

If the discoveries of those equations are traced in the modern mathematics, it would remain as an incomplete task because nothing much is known on the history of this topic. Almost all these equations/ formula in the theoretical and applied field, were known to Indian mathematicians. It is important to remember that many more such rules and applied mathematical problems are available in the books mentioned above. The above set of examples and rules have been quoted to show that this subject was of great interest to our forefathers millennia ago.

Determination of unknown value from sums, products, etc.,

Determination of specific values from sums/products/ ratios/differences of two or more numbers is common. Simple and well defined procedures were known to ancient Indians for finding the solutions. Neelakanta Somayaji in Tantra samgraham (1500 AD) has given rules under the subtitle, Dasa prasnotharam (Answers for ten questions). He says:

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राश्योरयोगोभिघातो वर्गयोगस्तदन्तरम् ।
एषु द्वाभ्याम् दशविधं राश्योरानयनम् भवेत् ।।

Raasyorayogobhighatho vargayogasthadantharam
eshuduaabhayaam dasavidham raasyoraanayanam bhaveth

For finding out the unknown values in a problem, when the sum, difference, square and their differences, etc., are given, there are ten different methods. In Bhaskaracharya II's Lilavati (page 86) an example of this problem is given.

ययोर्योगशतं सैकं, वियोगः पञ्चविंशतिः ।
तैराशी वद मे वत्स वेत्सि संक्रमणं यदि ।।

Yayoryogasatham saikam, viyoga: panchavimsathi:
thairasee vada me vatsa retsi sankaramanam yadi

When two numbers are added, it gives 101, and subtracted the result is 25. Tell me boy what are the numbers?

If the numbers are say, x and y. Then, $x+y = 101$ and $x-y = 25$. When these two are added, value for 2x will be obtained. Half of that will be, one of the unknown numbers (x), which is 38 and the other (y) can be determined by substituting the value for x. This rule is stated in different way in Vedaganitham [11].

वर्गान्तरात् योगभक्तो भेदस्तेनापि पूर्ववत् ।

Vargaantharaath yogabhaktho bhedasthenaapi poorvavath

Half of the sum of the sum and difference will give one value and half of the difference of sum and difference of the values will give the other

I.e if the sum is A and the difference is B, then ½ (A+B) gives x and ½(A - B) gives y where x and y are two unknown numbers. Similar problem when difference and product of two numbers are known, is given by Bhaskaracharya I in his Bhashya for Aryabhateeya (113.1)

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संवर्गोष्टौ दृष्टो व्यक्तं तत्रान्तरं भवेद्द्वितयं ।
अष्टादशके मुनयो गुणकारौ तौ तयोर्वाच्यौ ।।

Samvargoshtow drushto vyaktham thathraantharam bhavedvithayam
ashtaadasake munayo gunakaarow thow thayorvaachyow

The product of two numbers is correctly seen to be 8; their difference is 2. For two other numbers the product being 18 and difference is 7. Tell, the numbers multiplied in the two cases (i.e all the four numbers)

By assuming the numbers, as x, y, ... etc., and following the procedure, answer will be obtained. Product of numbers from the sum and sum of the squares can be determined according to the method given by Aryabbatta I (Aryabhateeya 2-23)

सम्पर्कस्य हि वर्गाद् विशोधयेदेव वर्गसम्पर्कम् ।
यत्तस्य भवत्यर्धम् विद्याद् गुणकारसंवर्गम् ।।

Samparkasya hi vargaath visodhayedeva vargasamparkam
yatthasya bhavathyardham vidyaad gunakaarasamvargam

From the square of the sum of two factors, subtract the sum of squares. One half of that should be known as the product of the two factors.

If a and b are two factors, then $a \times b = \frac{1}{2}\ (a+b)^2 - (a^2+b^2)$

Mishra and Singh [52], say that the credit of finding a solution for the first degree indeterminate equation, by a method called Kutta (literally means pulverizer) by breaking into smaller fragments by means of continued division, goes to Aryabhatta I. The method resembles the continuous fraction process developed by Euler in 1764. More than 1250 years before Euler, Aryabhatta I could find out a solution for the indeterminate equations. Higher levels of applications have been achieved by many commentators of Aryabhateeya later. Hardikar[53], has also proved that the solutions of indeterminate and first order

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equation were discovered by Indians, millennia ago.

A number of applied problems are given on this subject in many Sanskrit books. A few examples are given below from Sreedharacharya's Patiganita (73, 74)

मुद्गानां कुडवाः सप्त लभ्यन्ते नवभिः पणैः
पणेन कुडवस्यार्धं तण्डुलानामवाप्यते ।
ततः पणत्रयं सार्धं गृहीत्वाऽऽशु वणिङ्मम
तण्डुलानां प्रयच्छांशं मुद्गानां च द्विसङ्गुणम् ।।

Mudgaanaam kudavaa. saptha labhyanthe navabhi: pane.
panena kudavasyaardham thandulaanaamavaapyathe
thatha: panathrayam saardham gruheethvaa fasu vaningmama
thandulaanaam prayacchaamsa mudgaanaam cha dvisangunam

7 kudavas (unit of measurement) of mudga are obtained for 9 panas and ½ kudava of rice is obtained for one pana. Then O! merchant take 3½ panas and quickly give me one part of rice and two parts of mudga.

Finding out the quantity per unit pana is to be followed for the answer. The quantity of mudga is (49/32) and rice (49/64).

Bhaskaracharya II has given a problem of the simple order intermediate equation for finding out the unknown number from a final value when the initial number has undergone a 'series of processsing' in Lilavati (77-2)

अमलकमलाशशेस्त्र्यंशपंचाशषष्ठैस्त्रिनयनहरि सूर्या येन तुर्येण चार्या ।
गुरुपदमथषड्भिः पूजितं शेषपद्मैः सकलकलसंख्यां क्षिप्रमाख्याहि तस्य ।।

Amalakamalaasasesthrayamsa panchaashashtai
sthrinayanahari soorya yena thuryena chaaryaa
gurupada mathashatbhi: poojitham seshapanchai.
sakalakalasankhyaam kshipramaakhyaahi thasya

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One third of the total lotus flowers were offered for performing pooja to Sankara, 1/5 to Vishnu, 1/6 to Surya, 1/4 to Devi and remaining 5 to Guru. What was the total number of lotus flowers?

Solution for this problem can be obtained by assuming that the number of total lotus is x and taking the sum of all fractions. I.e $1/3\,x + 1/5x + 1/6x + 1/4x + 5 = x$. From this the value of x can be calculated. Another type of similar problem for determining a number is given by Bhaskaracharya I in Aryabhateeyabhashya (124.1)

द्विगुणं रूपमेतम् पञ्चविभक्तम् त्रिताडितं भूयः ।
द्व्यूनं सप्तविभक्तम् लब्धम् रूपं कियत् भवेत् पूर्वं ।।

Dviguṇam roopametham panchavibhaktham thrithaaditham bhooya:
dvyoonam sapthavibhaktham labdham roopam kiyath bhaveth poorvam

A number is multiplied by 2, increased by 1, divided by 5, multiplied by 3, then diminished by 2 and divided by 7, the result is 1. Say what is the number?

Answer can be derived by tracing back method, stepwise assuming the unknown number as x. Yet another example given in Bhaskarabhashya (133 1).

पञ्चभिरेकं रूपं द्वे रूपे चैव सप्तभागेन ।
अवशिष्यते तु राशिर्विगण्यतां तत्र का संख्या ।।

Panchabhirekam roopam dve roope chaiva sapthabhaagena
avasishyathe thu raasiviganyathaam thathra kaa sankhyaa

A number leaves 1 as the remainder when divided by 5 and, 2 when divided by 7. Calculate the number.

Using intermediate equation method, solution for this problem could also be found out.

The method of equating two parameters of which one is

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known and the other is unknown, is also adopted in mathematics. Bhaskaracharya I has equated two persons who are equally rich, having two items in different quantities and told to find out the price of the unknown item. (Aryabhateeyabhashyam I, 127 and 128)

कुङ्कुमपलानि चाष्टावेकस्य धनस्य रूपका भवति
द्वादशपलानि विद्यावन्यस्य धनस्य रूपकास्त्रिंशत् ।
तुल्यार्धेण च क्रीतं कुङ्कुमं द्वाभ्यां कियत् पलार्धेण
इच्छानितत्र बोद्धुं मूल्यं वित्तं च तुल्यमेव तयोः ॥

Kunkumapalaani chaashtaavekasya dhanasya roopakaa bhavathi
dvaadasapalaani vidyaavanyasya dhanasya roopakaasthrimsath
thulyaardhena cha kreetham kunkumam dvaabhyaam kiyath palaardhena
icchaanithathra boddhum moolyam vittham cha thulyameva thayo:

A certain person has 8 palas of saffon and money amounting to 90 rupakas, another person possesses 12 palas of saffron and 30 rupakas (and two are equally rich). If two persons have bought the saffron at the same rate per pala, I want to know the price of one pala of saffron and also equal wealth possesed by the two, (equivalent in rupakas)

This problem has to be solved using a simple equation $8x+90=12x+30$ from this x (price of saffron) can be obtained and substituting the value for x, total wealth can be determined.

In yet another problem given in Bhaskarabhashya (128 4) his question is asked

नव गुलिका सप्त(च) रूपकसमास्त्रयाणा (तु) गुलिकानां ।
त्रयोदशानां च रूपकाणां तदा किं गुलिकामूल्यम् ॥

Nava gulikaa saptha cha roopakasamaasthrayaanaam (thu) gulikaanaam
thrayodasaanaam cha roopakaanaam thadaa kim gulikaa moolyam

9 gulika and 7 rupaka are equal to 3 gulika and 13 rupaka.

what is the price of one galika? (the answer can be determined through the same method followed above)

In Patiganita (52 u) an important problem on the quality of metallic alloy is given, in respect to the colour of product. From this the composition should be found out. It is like finding out the average of many averages.

हेमगुणवर्णयोगे हेमैक्यहृते भवेद्वर्णः ।

Hemagunavarnayoge hemaikyahruthe bhaved varna:

The sum of the products of weight and varna of several pieces of gold being divided by the sum of the weight of the pieces of gold, gives the varna of alloy

I.e n pieces of gold of weight wa, wb, wc, wd, ...wn and varnas va, vb, vc, vd.... vn, then varna of the alloy v = (wa va + wb vb + wc vc + ...wn vn)/ (wa + wb + wc + ...wn)

A problem of significance is given in Lilavati by Bhaskaracharya II (p.129 ex.13-1)

ये निर्झरा दिनदिनार्ध तृतीय षष्ठैः संपूरयन्ति हि पृथक् पृथगेव

मुक्ताः । वापीं यदा युगपदेव सखे विमुक्तास्ते

केन वासरलवेन तदा वदाशु ।।

Ye nirjaraa dinadinaardha thrutheeya shashtai:
sampoorayanthi pruthak pruthakeva mukthaa:
vaapeem yadaa yugapadeva sakhe vimukthaasthe
kenavaasaralavena thadaa vadaasu

By opening 4 inlets separately, one pond gets filled respectively within 1, ½, 1/3, and 1/6 days. If all the four inlets are opened together, how much time (in fraction of the day) is required to fill the pond.

Equations of the higher order:

Higher order equations, including quadratic equations were handled by Indian mathematicians. Solution for arriving at the

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answer of quadratic equation is given in "Vedaganitam" (Page 35 , 36 [33]).

चतुराहतवर्गसमैः रूपैः पक्षद्वयं गुणयेत् ।
अव्यक्तवर्गं रूपैर्युक्तौ पक्षौ ततो मूलम् ॥

Chathuraahatha vargasamai: roopai pakshadwayam gunayeth
avyaktha varga roopairyukthow pakshow thatho moolam

Add on both sides of an equation, 4 times the unknown value. Again add on both sides the square of the unknown value and take the square root. (This summarised procedure can be mathematically presented as follows:)

$ax^2 + bx = c$ Multiply both sides with 4a
$4a^2x^2 + 4\,abx = 4ac$ Add b^2
$4a^2x^2 + 4abx + b^2 = 4ac + b^2$
$(2ax + b)^2 = 4ac + b^2$ I.e $2ax + b = \sqrt{(4ac + b^2)}$
from the above, x can be calculated, says the rule.

Sreedharacharya and Bhaskaracharya I have discussed many examples on the application of the quadratic equations. One each from these two authors are given. (Patiganita example 100)

वानरकुलत्रिभागः स्वत्र्यंशसमन्वितः सरः प्रययौ ।
मूलं च पिपासति द्वौ चूततले स्थितौ शेषौ ॥

Vaanarakulathribhaga: swathryamsa samanvitha: sara: prayayow
moolam cha pipaasathi dwow choothathale sthithow seshow

One third of a troop of monkey with one third of itself has gone to the tank; the square root of the whole troop is afflicted with thirst, and the remaining 2 monkeys are sitting under the mango tree. What is the total number of monkeys?

The problem can be written as follows:

$1/3a + 1/9a + \sqrt{a} + 2 = a.$

This equation is quadratic in nature and the solutions can be found out as explained before using the standard method.

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Bhaskaracharya's Lilavati (page 95 example 1) gives the following exercises

बाले मरालकुलमूलदलानि सप्त तीरे विलासभरमन्थरगाण्यपश्यम् ।
कुर्वञ्च केलिकलहं कलहंसयुग्मं शेषं जले वद मरालकुलप्रमाणम् ॥

Bale maralakula mooladalaani saptha theere vilaasabhara mantbara gaanyapasyam kurvancha keleekalaham kalahamsayugmam sesham jale vada maraalakula pramaanam

I saw that one half of 7 times of the square root of the total number of swans were slowly moving away in the river. Remaining 2 are playing in water. What is the number of total swans? (equation: $7/2 \sqrt{a} + 2 = a$)

Mahaveeracharya (815 AD) in his Ganita sarasangraha (4-41) has also given an interesting example, the translated version of which is given below.

"Among the total elephants, 1/3rd of them and three times the square root of the remaining are in the valley. One male with three female elephants is in the river. Find out the total number of elephants". (Equation : $1/3a + 3\sqrt{a} + 1 + 3 = a$)

All these authors have given in detail the rules and methods through which it should be worked out. It is noteworthy here that the commentators of the original works have gone deep into various aspects of the equations and have expanded the theoretical and practical scope, by incorporating and updating the knowledge.

The credit of discovering binomial theorems and their application should go to Indians. The theorem in its modern and simplest form can be written as $(a+b)^n = a^n + \ldots + b^n$ i.e n is positive integer. This form of writing the equation was done by an Indian scholar, Halayudha of 10th century AD[31]. Higher level binomial equation was arrived at by him based on the principles of syllables, given in Chanda sutra of Pingalacharya (200 BC). He used the examples to show not only the theoretical capability

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of the ancient mathematicians but also the correct method of its application in distinguishing the sound intensity. Derivation was arrived at by him as follows: If a three syllabic Madhya Chanda based on guru and lakhu sounds were followed, then variation of guru and lakhu sound will be on the following pattern: 3 guru sound occur once, 2 guru and 1 lakhu occur thrice, 1 guru and 2 lakhu sounds occur thrice, 3 lakhu occur once. The equation can be derived easily. If guru is g and lakhu is 1 then,

$(g+1)^3 = g^3+3g^2 1+3g 1^2+1^3$. This equation is the same as $(x+y)^3$. Similarly for finding the pratishta Chanda, in the Chanda sastra of Pingalacharya, the following equation can be indirectly applied in this form: $(g+1)^4$ which is expanded as $g^4+4g^3 1+4g^2 1^2+4g 1^3+1^4$ i.e 4 guru sound occur once, 3 guru and 1 lakhu occur four times, 2 guru and 2 lakhu occur four times, 1 guru and 3 lakhu occur four times and 4 lakhu occur once.

More information was also known on the higher levels of equations, as stated by Madhavacharya in Kriyakramakari. Here, Madhavacharya quoted another famous mathematician Chitrabhanu who has given the following equations [54].

$$(a^3 - b^3 - (a-b)^3) / 3(a-b) = ab$$

$$(a^3 - b^3 - (a-b)^3 / 3\,ab = a-b$$

$$(2\,(a-b) + (a-b)\,/3\,(a^2+b^2) = a-b$$

Summing up the descriptions on these mathematical equations and rules one can understand that the algebra of even higher order were used in India centuries before the present state of development in the subject. The credit of discovery of the quadratic equation and all other binomial equations have been given to the Europeans Apianus (1527 AD), Stifel (1544 AD), Acheubel (1545 AD), Tartagalia (1556 AD) and Bombelli (1572 AD)[55]. But is was the Indian scholars who gave the foundations for the binomial theorems with theoretical and applied back ground and common applications.

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Geometrical Studies

A well defined chronlogical pattern can be seen in the development of Geometry in India. Mohanjo daro, Harappa, Lothal and dozens of other archeologically important sites (and some more are being excavated) have shown that the constructions have definite geometrical patterns[56]. They are triangular, quadrilateral, circular or imposed structures and all of them were constructed by following refined geometrical applications. Some of these structures have a mathematical history, of not less than 5000 to 6000 years.

However, from the available literature in Sanskrit, the Sulba sutras stand first. These books deal with the ritual in the Sroutha sutra and at the end of the Sroutha sutra comes the Sulba sutras. But the Sulba sutras have a definite Sanskrit style and compositional structure of the Vedic Sanskrit language. The prominent among the Sulba sutras are Bhoudhayana, Apastamba, Katyana and Manava Sulba sutras. Even though the periods of these books are generally fixed at the beginning of first millennia BC, some historians estimate the age to be in 2 and /or even 3rd millennia BC[57]. Even if we fix the period to 1000 BC. It is far older than the defined period of Greek mathematicians, which falls between 500 - 200 BC [58]. The selected quotations from Sulba sutras can give the modern geometrical concepts developed in ancient India.

Bhoudhayana Sulbasutra : This is chronologically estimated to be the first among all the Sulbasutras. It may be around 1000BC and this Sulbasutras got the present form.

Quotations, without detailed explanations are given below, as they are self-explanatory, geometrical informations.

व्यायाममात्री भवतीति गार्हपत्यचितेर्विज्ञायते ।
चतुरश्रेत्येकेषां परिमण्डलेत्येकेषां ।

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Vyaayaamamaathree bhavatheethi gaarhapathyachither vijnayathe chathurasrethyekaashaam parimandaleethyekeshaam

According to tradition the Garhapatyaygi fire altar has the measure of area one vyayama. It is a square according to one tradition and a circle by another (II.61-63). This is an example to show the emergence of the geometrical background in Sulba sutras.

प्रौगचितिम् चिन्वितेति । *Prowgachithim chinuithethi*

For prouga chiti a fire altar in the form of isosceeles triangle is to be constructed (III. 161): shows how theorems and equations related to triangles emerged in the ritual books.

उभयतः प्रौगम् चिन्वितेति । *Ubhayatha: prougam chinvithethi*

Ubhaya prouga chiti, a rhombus with two isoceeles triangles is to be constructed (III. 172) (relation between the rhombus and isosceles triangles are dealt with)

यावानग्निः सारत्निपादेशाष्टवदुभयतः प्रौगम् कृत्वा ......

Yaavanagni saarathni paadesaashta vadubhayatha: prougam kruthva...

(For yavanaagni a rhombus equal in area to two aratnis and one pradesa (1½ purusha in area) is laid (III 173) (method to generate a geometrical figure from its area).

Different types of bricks are described as follows. Bricks with one side equal to one fifth of purusha (Purusha is a measurement equal to the height of the yajamana) Quarter bricks bounded four sides ½ pradesa (6 angulas. average diameter of the finger is one angula), 1½ pradesa (18 angulas), 1 pradesa and $\sqrt{2}$ pradesa. Adhyardha bricks bounded four sides ½ vyayama (48 angulas) 1 aratni (24 angulas) and $\sqrt{2}$ aratni. These make 6 types of bricks (12.7)

This example gives a lot of information on the use of square root, relation among various measurements, manufacture of

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bricks, having different shapes and sizes, etc. and also the approach to arrive at various structures by including many parameters, as a branch of ganita.

समचतुरश्रस्याक्ष्णयारज्जु द्विष्टावतीं भूमिं करोति ।

Samachathurasrasyakshnayaa rajju dvishtavathim bhoomim karothi

The diagonal of a square produces double the area of the square (1.45) a rule to be followed in the construction of chiti. It is a geometrical theorem.

दीर्घचतुरश्रस्याक्ष्णयारज्जुः पार्श्वमानि तिर्यन्मानि

च यत्पृथग्भूते कुरुतस्तदुभयं करोति ।

Deerghachathurasrasyakshnayaarajju: paarsvamaani thiryanmaani cha yatpruthakbhoothe kuruthasthadubhayam karoti

Areas produced separately by the length and breadth of rectangle together equal to the area of the (square) produced by the diagonal (1.48) (gives the directions, to make the geometrical figures under specified rules:)

तासां त्रिकचतुष्कयोर्द्वादशिकपञ्चिकयोः पञ्चदशिकाष्टिकयोः

साप्तिकचतुर्विंशतिकयोः द्वादशिकपञ्चत्रिंशिकयोः

पञ्चदशिकषट्त्रिंशिकयोः इत्येतासूपलब्धिः ।।

thaasaam trika chathushkayordvaadasikapanchikayo: panchadasikaashti kayo: saaptikachathurimsathikayo: dvaadasika panchathrimsathikayo: panchadasikashad-thrimsikayo: ithyethaasoopalabdhi:

This is observed in rectangles having sides 3 and 4 (=5), 12 and 5 (=13), 15 and 8 (=17), 7 and 24 (=25), 12 and 35 (=37) and 15 and 36 (=39) (1.49).

Here, with examples Pythagorus theorem (Bhoudhayana theorem) is explained as: when two squares are constructed on the length and breadth of a rectangle, the square obtained from

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the diagonal of the rectangle will have sides without fractions- as whole numbers given in parenthesis-for the rectangles having above set of lengths and breadths.

Apastamba Sulbasutra:

This appears to be of later origin than Bhoudhayana Sulbasutra, according to some historians. More descriptions and significant information on the structure of altars and geometrical figures are given in Apastamba Sulba sutra. Some methods and geometrical structures given in the Boudhayana sutras are also repeated here with minor variation in language style. A few important and relevant quotations are given below. Some of them are improvements and addenda to earlier writings.

For different agni (yajna ritual) altars with varying areas are to be constructed. (8.3)

एकविध: प्रथमोऽग्निर्द्विविध: द्वितीयास्त्रिविधस्तृतीय:
त एवमेवाद्यन्त्यैकशतविधात् ।।

Ekavidha: prathamognirdvividha, dvitheeyarthrividha sthrutheeya: tha evamevadyanthyai sathavidyaath

The first agni is one fold (1½sq purusha) in area, the second 2 fold (2½ sq. purusha) and third 3 fold (3½sq. purusha) and so on. In this way one has to go upto hundred and one fold agni (with the sacrifical altar having area 101½ sq. purusha) - (Even for the same type of yajna, when repeated year after year by same yajamana, the size of the altars is to be increased. This increase in size has a mathematical relation. Hence this also emphasises application of geometry at various instances)

पुरुषमात्रेण विमिमीते इति विज्ञायते ।

Purushamaathrena vimimeethe ithi vijnayathe

According to tradition all the measurements are to be made with purusha unit i.e to be measured with bamboo (8 7).

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This shows that there was a definite mechanism for the standrad of measurements and the measuring scale.

पादमात्र्यो भवन्ति आरत्निमात्र्यो भवन्त्युर्वस्थिमात्र्यो
भवन्त्यणूकमात्र्यो भवन्तीति विज्ञायते ॥

Paadamaathryo bhavanthi aarathnimaathryo bhavanthyoo ruasthimaathryo bhavantheethi vijnayathe

Other measurement units to find out the area are pada, aratni,urvasti, anuka, etc. (11.2) They are related as follows.

चतुर्भागीयमणूकं पञ्चमभगीयारत्नी ततोर्वस्थि ॥

Chathurbbaageeyamanookam panchabbaageeyaaratnee thathoruasthee

Anuka is 1/4 of a purusha, aratni is 1/5 of a purusha, and so is urvasti which is 1/6 of purusha.(11.3) (It gives a clear picture of the relationship between the different units of measurements).

Using the areas, diagonals, or one of the side, the geometrical structures were constructed When area of the circle is given, it has to be constructed after finding out the diameter. Hence the method of finding out the diameter was expected to be known. The same is the case for all other structures. Constructing square from its diagonal, rhombus from diagonal, square inscribed circle and so on... Rules are clearly given for this purpose. Similarly from the mesurements of the altars and the number of bricks required to construct the same, bricks are to be designed ~~and made~~ based on the geometrical knowledge. No broken or smaller bricks can be used for final adjustment in construction purposes. Hence the brickmaker is expected to have a perfected geometrical back ground of high standard, so that the person involved in the construction of altars can extrapolate and intrapolate the measurements of bricks and their sizes. Some examples on this line are also given below.

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यावानग्निः सारत्नीप्रदेशाष्टावर्ति भूमिं परिमण्डलं
कृत्वा तस्मिंश्चतुरश्रम् अवदध्यावत् सम्भवेत् ।।

*Yaavaanagni saarathnee pradesaashtasathim bhoomim parimandalam
kruthvaa thasmimschathu rasram avadadhyaavath sambhaveth*

A circle of area equal to that of the fire altar with 2 aratnis and one paradesa is made and the largest possible square is inscribed in it (12 10) (The knowledge on the construction of a circle, after calculating the diameter from the area is the first step Then a square, suitably fixing in the circle, is to be inscribed after calculations). Here measurements of the structures are not crude or simple. It is complex too and an approximation will not work.

द्विपुरुषं पश्चादर्धपुरुषं पुरस्ताच्चतुर्भागोनः पुरुष आयामो/
ष्टादशकरण्यौ पार्श्वयाष्टः पञ्चदश परिगृह्णाति तत्पुच्छम् ।।

*Dvipurusham paschaadardhapurusham purasthaachhathur
bhagona. purusha: aayamo/shtadasa karanyow paarsvayoshta:
panchadasa parigruhnaathi thatpuccham*

(An area bounded by length of) 2 purushas on the western side, 1½ purusha on eastern side, √18 on each of other two sides and having height 3/4 purusha can accommodate 15 bricks (19.1)

Hence size of the bricks has to be calculated from the given data of the number of bricks permitted in building this altar.

षोडशीम् चतुर्भिः परिगृह्णीयात् अष्टमेन त्रिभिरष्टमैश्चतुर्थेन
चतुर्थ सविशेषेणेति ।।

*Shodaseem chathurbhi: parigruhneeyaath ashtamena
thribhirashtamaischathurthena chathurtha savisesheneth*

The bricks to be made by four sides having the measurements. 1/8, 3/8, 1/4 and √2/4 purusha (19.2)

This obviously stands out as an example on the complicated

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bricks making with the background of geometry. Sutra (1 5) says:

चतुरश्रस्यक्ष्णयारज्जुर्द्विष्टावतिं भूमिं करोति समस्य द्विकरणि ।।

Chathurasra syakshnayaarajjurdvishtavathim
bhoomim karothi samasya dvikarani

The diagonal of the square produces double the area (of the square) and it is √2 of the side of the square.

It is important to note here that the method for the determination of √2 is given in Boudhayana sulba sutra, for the first time in mathematics (I.61, 62) This is also explained under Katyayana sulbasutra.

प्रमाणं तृतीयेन वर्धयेत्तच्च चतुर्थेनात्मचतुस्त्रिंशोनेन सविशेषः ।।

Pramaana thrutheeyena vardhayethaccha
chathurthenaathsachathusthrimsonena savisesha:

Increase the measurement (side of a square) by third and this third by it's own fourth less the thirty fourth part of that fourth. This is the value with special quantity in excess for √2.

Apastamba sutra (5.3 to 5.6) lines give measurements of sides of rectangles which can give diagonal measurements in whole numbers. This is an example of applied knowledge.

त्रिकचतुष्कयोः पञ्चिकक्ष्णया रज्जुः। द्वादशिकपञ्चिकयोः त्रयोदशिकक्ष्णयारज्जुः । पञ्चदशिकाष्टिकयोः सप्तदशिकक्ष्णयारज्जुः ।।

Thrika chathushkayo: panchikakshnayaa rajju:
dvaadasika panchikayo. thrayo dasika kshnayaarajju:
panchadasikaashtikayo: sapthadasikakshnayaarajju:

The diagonals of rectangle with sides 3 and 4 is 5, 12 and 5 is 13, 15 and 8 is 17.

Transforming one figure into another is frequently discussed in the book.

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मण्डलम् चतुरश्रम् चिकीर्षन् विष्कम्भं पंचदशभागान्
कृत्वा द्वावुद्धरेत् त्रयोदशावशिष्यन्ते सानित्याचतुरश्रम् ॥

Mandalam chathurasram chikeershan vishkambham panchadasa bhaagaan kruthvaa dvaavuddhareth thrayodasaavasishyanthe saanithyaachathurasram

To transform a circle into a square, diameter is divided into 15 parts, 2 of them are removed, leaving 13 parts. This gives the approximate side of the square (3.3)

This however cannot give a good result. In Manava Sulba sutra better approximations are given for this calculation, which is referred elsewhere also.

The following important rules are also given in Apastamba Sulbasutra: Rule 3.2 is for transforming a square into a circle, 3 1 is for transforming a square into reactangle, 12.1 which has been stated earlier is inscribing a square in a circle of known area. Similarly inscribing different figures in other geometrical figures is given for the construction of required/suggested types of altars.

Katyayana sulba sutra: This book gives a variety of structures for altars: Katyayana Sulba sutra (4-I) says:

द्रोणचिद्रथचक्रचित्कंकचित्प्रौगचिदुभयतः प्रौगः समूह्यापुरीषा इत्यग्नयः ॥

Dronachidrathachakrachithkankachith prougachidubhayatha; prowga: samoohyaapureeshaa ithyagnaya:

The altar is of the shape of a trough, chariot wheel, falcon, triangle, rhombus and a kind of pot in the shape of wheel

In all these structures the size, sides, shape of the bricks and their numbers are specified. Hence the construction procedure have to be based on the geometrical information Methods of finding the square roots are also given Stanza 2.9 describes the method for finding the diagonal of a square.

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करणीं तृतीयेन वर्धयेत्तच्च स्वचतुर्थेनात्मचतुस्त्रिंशोनेन
स विशेष इति विशेषः ॥

Karaneem thrutheeyena varddhayethachha svachatheerthenaa thmachathu sthrimsonena sa visesha ithi visesha·

The measure of the side is to be increased by one third of its value, again its own one fourth, less the 34th part (of that of fourth).

This is the diagonal of a square, whose side is the measure and this is approximate, which is also explained in other suthras. I.e if the side is a the diagonal a+a/3+(1/3x4) - 1/3x4x34. This can give a fine approximation for the value of √2 a = 1.4142156a. . The value for √2 = 1.414212 The modern and ancient values are almost the same. This value and method are reproduced here from Bhoudhayana Sulbam. Thibaut[39] says about this in his observation "... thus it is clear that the ancient Hindus have attained remarkable degree of accuracy in calculation of the approximate value for √2"

A detailed description on the construction of yajnasala (ritual hall) is given in the Manava sulbasutra (3.1-3 3). Infact this description gives the beauty and rule of the geometrical contents in the sulba sutra.

प्राग्वंशम् दशकम् कुर्यात्पत्नीशालां चतुःशयम् । प्राग्वंशतृषु वेद्यन्तो वेद्यन्तात् प्रक्रमे सदः । नवकम् तु षडो विद्याच्चत्वारः षडशोन्तरम् । चत्वारिंस्त्रिक हविर्धानामर्धदशाष्टादन्तरम् । पादम् यूपवतेमित्वा शेषमौत्तरवेदिकम्। अग्नीध्रम् षडारत्न्येव षड्त्रिंशत्प्रक्रमा रज्जुः ।

Pragvamsam dasakam kuryaathpathneesaalaam chathu: sayam praagvam sathrushu vedyantho vedyanthaath prakrame sada: navakam thu shaddo vidyaachhathvara: shodasontharam chathvaaristhrika havirdhanaamardhadasaashtaadantharam paadam yoopavathemithvaa seshamowttharavedikam agneedram shadaarathnyeva shadthrivimsathprakramaa rajju:

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The sacrifical hut pragvamsa occupies (on the ground a square area of) side 10 aratni, the hut for the wife (patnee sala) (a square) of 4 aratnis, end of the (maha) vedi is at a distance of 3 prakrama from the pragvamsa, and the hall sadas is one prakrama away from the western end of the mahavedi. The (praci) sadas is 9 prakramas The havirdana (a square) of 12 prakramas is 4 prakramas from the sadas and 10½ prakrama from the yupavata. One pada is allowed for the yupavata and the remaining belongs to the uttaravedi. The agnidra hut (a square) of 6 aratnis, the chord measures 36 prakramas.......

Manava Sulba sutras: Almost all the mathematical descriptions given in above three can be seen in this Sulbasutra also. It is very important to note that many fundamental knowledge which are not described in other Sulbasutras have found their place in Manava sulbasutra. An example is the method for finding the volume of a structure given in the line 10.9

आयम बाहुम् निक्षिप्य विस्तारस्तु तथापृथक्
सो/ध्यर्धम् गुणयेद्राशिम् ससर्वगुणितो घनः ।।

Aayama baahum nikshipya visthaarasthu thathaapruthak
sondhyardham gunayedraasim sasarvagunitho ghana·

Multiply the length with the breadth separately and that again by the height This always gives the result in cubic measure.

This appears to be the first ever seen in a literature for finding out the volume of a structure. These lines are written centuries before Archimedes, who found out the volume of an object by dipping it in a water tank. A few more quotation from the Manava sulba sutra on the circular structure are given, (11.13)

विष्कम्भः पञ्चभागश्च विष्कम्भस्त्रिगुणश्च यः
सा मण्डल परिक्षेपो न बालमतिरिच्यते ।।

Vishkambha: panchabhaagascha vishkambhastrigunascha
ya: saa mandala pari kshepo na vaalamathirichyathe

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The fifth part of the diameter added to three times the diameter gives the circumference of the circle. Not a hair is left over.

For circle 3.14 x diameter is the circumference. As per Manva sulbasutra it is 3.2 x diameter, which is perhaps the first approximation for ∏ . The descriptions given in Apastamba sulba sutra (3 3-given earlier) to convert a circle into a square was only an approximation. More refined rule is given here (11.14)

दशधा चिन्द्य विष्कम्भम् त्रिभागानुद्धरेत्ततः
तेन यच्चतुरश्रम् स्यान्मण्डले तदपप्रथिः ।।

Dasadhaa chindya visbhambham thribhaagaanuddharethbatha:
thena yachhathurasram syaanmandale thadapaprathi:

Divide the diameter of a circle into ten parts and leave out three parts. The square drawn with this (as side) and placed within the circle just projects.

The measurements of the bricks given in rule 14.21 can throw light, not only on the geometrical accuracy followed in brick manufacturing but also the ceramics techniques followed during the period.

चत्वारि कराण्यन्येषां त्रिचतुर्थेन कारयेत् नवभागा
अक्ष्णार्धक्ष्णाः पञ्चकोणाः च भागशः

Chathvaari karaanyanyeshaam thrichathurthena kaarayeth
navabhaagaa akshnaardhakshnaa: panchakonaa: cha bhaagasa:

Four kinds of bricks were used with one third and one fourth (of a purusha) measurement. These are one ninth of the original (40, 40), triangular (30, 30, 30 √2), half triangular (15 √2, 30, 15 √2) and five cornered bricks (15 √2, 15 √2,15, 30, 15).

From the above selected quotations, a lot more information can be elucidated. It is obvious that the geometrical background

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was excellent, in sulbasutras They acted as the strong foundation for building up the modern mathematics.

Even the actual proof of Pythagorous theorem was given in Greek only in 300 BC by Euclid, says Burk [60]. Proclus (460 BC) has made it clear that it is not an original contribution of the Greeks, as also proved by Mishra and Singh Hence why not the mathematician think of renaming the theorem as Bhoudhayana theorem!

Little more information if added to the above, the exact practical form of Bhoudhayana theorem is stated to have been known in Brahmanas (explanation of ritual part of the Vedas-also known as karmakanda) during 2500 BC. Satapata Brahmana, Taitireeya Brahmana and Garga samhita also contain this information, Seidenberg [61] has shown that Taitireeya samhita describes not only the algebraic or Computational aspects of this theorem but also geometric or sonstructive aspects which was not known to any others. (This is quoted from Seidenberg 1983, the geometry of Vedic rituals, in Agani, Vol II. Frits Staal, Asian Humanities press, Berkeley, page 125-126). Burk has concluded that the theorem was known to Indians with all its proofs, in the far past Pythagorus.

Many studies have thrown light on the geometrical contents in these Sulbasutras on which modern mathematicians could arrive at important conclusions, thousands of years later. Construction of a line perpendicular to other is derived from Katyayana Sulbasutra (rule 1.4 and 1.5). Similarly construction[62] of squares having areas either sum of two other squares, or differences of areas of two squares are also given in Apastamba sulbam 2.4, Boudhayana sulbam 2.1 and 2.2 and Katyayana sulbam 2.13 and 3.1

Triangles : Details on the triangles are given in the Sulbasutras.

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They contain measurements of sides, hypotenuse, areas and also the relations among these parameters. Inscribing other geometrical structures in the triangle and vice versa are also explained, including the methodology for doing so. Bhakshali manuscripts contain a lot of information on the geometrical figures including the triangles. Bhaskarachraya I has given the examples of application of the rules (given by Aryabhatta I) on the lamp and shadow calculations. Bhaskarabhashyam to Aryabhateeyam gives the examples with details on triangles (92.3).

यष्टिप्रदीपमूलात् पञ्चाशद्विवरसंस्थितः शङ्कुः ।
तस्यच्छाया पङ्क्तिर्वाच्यस्तस्मिन् कियान् दीपः ॥

Yashti pradeepamooolaath panchaasadvivuarasamsthitha sanku:
thasya cchayaa pangthiruaachyasthasmin kiyaandeepa:

The shadow of the gnomon situated at a distance of 50 angulas from the foot of the lamp post is 10 angula. Say what is the height of the lamp.

Height of the lamp post can be found out from length of the shadow and the length of the tip of shadow and lamp using Bhoudhayana theorem. Another interesting problem on the same line is about the bamboo triangle, given by the author in the same book (99.4)

अष्टादशकोच्छ्रयो वंशो वातेन पातितोमूलात् ।
षडगत्वाऽसौ पतितास्त्रिभुजं कृत्वा क्व भग्नः स्यात् ॥

Ashtaadasakocchrayovamso vaathena paathithomoolaath
shadgathvaavasow pathithaasthribhujam kruthvaa kva bhagna: syaath

A bamboo of height 18 cubits fell by the wind, it falls at a distance of 6 cubits from the root, thus forming a right triangle, where is the break?

This problem can be worked out as follows: The sum of

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lengths of hypotenuse and height is 18 cubit and the base of the triangle formed is 6 cubits. From this the height (x) can be calculated as $(18-x)^2-x^2=6^2$.

Same problem is repeated by Pruthudaka in his commentary on the Brahmasphuta siddhanta (XII. 41) and also in Mahavira in Ganitasara sangraha (VII.191, 192) Sreedharacharya, Bhaskaracharya I and II and many others have given problems of this type related to bamboo pole, lamp post-gnomon and lotus - river depth, etc Through this problem one can get the required exposure on the sound background of the ancient knowledge in the geometry of triangles.

Quadrilaterals: Brahmasphuta siddhanta (XII, 28) gives the equation for the diagonal quadrilaterals in relation with its sides: "The sum of the products of two pairs of sides about any diagonal divided by the sum of the products of pairs of the sides about other diagonal and the result multiplied by the sum of prducts of opposite sides give the square of the first diagonal". Similarly the second diagonal can also be calculated. First diagonal is x, (for sides a, b) and second y (for sides c, d) adjacent to them,

Then, $x^2 = (ab+cd)(ac+bd)/(ad+bc)$ And

$g^2 = (ad+bc)(ac+bd)/(ab+cd)$

Patiganita (110-111) gives a rule for finding out the area of various types of natural and aritifical geometrical figures in the following way:

आयतसमचतुरश्रे द्वित्रिसमभुजे विषमचतुरश्रम्
समविषमद्विसमभुजत्र्यश्रण्यथवृत्त चापे च ।
क्षेत्राणिदशैतानि हि फलमेषां साधयेत् स्वकरणे(न)
एतत् परिकल्प्याऽन्येषां गजदन्तनेमिपूर्वाणाम् ।।

Aayatha samachathurasre dvithri samabhuje vishamachathurasram
samavishamadvisamabhuja thrya sranyathavruthba chaape cha

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kshethraani dasaithaani hi phalameshaam saadhayeth svakarane(na)
ethath parikalpyaamyeshaam gajadanthanemi poorvaanaam

The rectangular quadrilateral, the equilateral quadrilateral, equibilateral quadrilateral equitrilateral quadrilateral....., the inequilateral quadrilateral, the equilateral triangle, the scalene triangle, isosceles triangle, circle and the segment of the circle are the ten primary plane figures; the area of these figures should be determined by applying their own rules. And by considering these, one should obtain the areas of other figures, such as an elephant, a buffalo, etc.

This is the method adopted in modern mathematics too, i.e dividing the total figure into geometrical sub figures/structures whose areas can be determined using standard equations. Then partial areas are added to get the actual area of the figure. Narayana, in Ganita kaumudi, has given the same 10 figures as mentioned in Patiganita. Mahavira in Ganitasara sangraha gives three varieties of triangles, five varieties of quadrilaterals and eight varieties of curvilinear figures including ellipse, conical, concave and convex and has given the detailed explanations.

Aryabhatta I (Aryabhateeya 2.6) gives the rule for finding out the area of any triangle as the product of the height and half of the base which is the same as that known now.

त्रिभुजस्य फलशरीरं समदलकोटि भुजार्धसंवर्गः।
Thribhujasya phalasareeram samadalakoti bhujaardha samvarga:

The area of a triangle is the product of the prependicular and half the base.

Bhaskara in his bhashya for Aryabhateeya gives problems for the calculation of areas of different types of triangles (55.1)

सप्ताष्टनवभुजानां क्षेत्राणां यत्फलं समानां तु ।
पञ्चश्रवणस्य सखे षड्भूसंख्याद्वितुल्यस्य ।।

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Saptaashta navabhujaanaam kshethraanaam yathphalam samaanaam thu
panchasravanasya sakhe shad bhoosankhyaadvithulyasya

Tell me the area of the equilateral triangle whose sides are 7,8 and 9 units respectively and also the area of isosceles triangle whose base is 6 units and lateral sides each 5 units. The above problems deal with equilateral and isosceles triangles. Similar problems on scalene triangles are also given by Bhaskara in his bhashyam (56.5)

कर्णस्त्रयोदश स्यात् पञ्चदशान्यो मही द्विसप्तैव ।
विषमस्त्रिभुजस्य सखे फलसंख्या का भवेदस्य ।।

Karnasthrayodasa syaath panchadasaanyo mahee drisepthaiva
vishamasthri bhujasya sakhe phalasankhyaa kaa bhavedasya

What is the area of a scalene traingle in which one lateral side is 13 units, other 15 unit and the base is 14 units.

This problem is also given in Brahmasphuta siddhanta (XII 21 ii), and was solved using the equation which is followed by the modern Mathematicians. I.e. finding out the semi perimeter of the triangle, and following the equation √S x (S - A) (S - B) (S - C), where S is the semi perimeter i.e half of the sum of all the three sides and A, B and C are the sides of the triangle. However Bhaskara I even though was a contemporary to Brahmagupta, has not adopted this equation in Bhaskarabhashya. He followed a different way which gives only an approximate value for the area of scalene triangles.

Polygonals: Bhaskaracharya II has given a very interesting information on the cyclic equilateral triangle and for Polygonals, in Lilavati (294-45,46 and 47)

त्रिद्व्यङ्काग्निनभश्चन्द्रैस्त्रिबाणाष्टयुगाष्टभिः वेदाग्निबाणखाश्वैश्च
खखाभ्राभ्ररसैः क्रमात् बाणेषुनखबाणैश्चद्विद्विनन्देषुसागरैः

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कुरामदशवेदैश्च वृत्तव्यासे समाहते खखखाभ्रार्क संभक्ते लभ्यन्ते
क्रमशोभुजाः वृत्तान्तस्त्र्यपूर्वाणां नवास्रान्तं पृथक् पृथक्

Thribdhyankaagninabha schandraisthri bhaanaa shtayugaashtabhi: vedaagni baanakhaaschaicha khakhaabhraabhrarasai: kramaath baaneshu nakha baanai schadvidvi nandeshu saagarai: kuraamadasavedaischa vruthhavyaase samaahathe khakhakhaabhraarka sambhakthe labhyanthe kramasobhujaa: vrutthaantha sthraya poorvaanaam navaasraantham pruthak pruthak

For cyclic equilateral triangle, cyclic square, cyclic equilateral pentagon,.... to cyclic equilateral nonagon, (cyclic figures having 3 to 9 sides with equal side measurements) their sides can be calculated respectively when diameter is multiplied separetely with 103923 (triangle) 84854 (quadrilateral) 70534 (pentagon), 60000 (hexagon) 52055 (septagon) 45922 (octagon) and 41031 (nonagon) and divided by 120000, the value will be the measurements of the sides of cyclic equilateral triangles to cyclic equilateral nonagon. Bhaskaracharya has given the example: If 2000 is the diameter of circle, equilateral geometrical figures inscribed inside that circle will have sides as follows:

Geometrical figure	Bhaskara's value	Modern value
Triangle	1732 + 1/20	1732.043
Square	1414 + 13/60	1414.211
Pentagon	1175 + 17/30	1175.5619
Hexagon	1000 + 00	999.996
Septagon	867 + 7/12	867.5799
Octagon	765 + 11/30	765.3636
Nonagon	683 + 17/20	683.85

Obviously the level of accuracy can be seen. No further discussion need be given for providing the capability of the ancient Indians on the structural knowledge of Polygonals This information on the relationship between the diameter and the

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side of the cyclic figures is based on the application of the mathematical equations; a very important contribution, made nine centuries ago.

Rectangles: Finding out the area of the rectangles was well known. An excercise is given by Bhaskaracharya I in Bhaskarabhashya (67.1)

अष्टौपञ्च च पङ्क्तिर्विस्तारे दैर्घ्यमप्यमीषां यत् ।
अष्टिर्द्वादश मनवो गणितं कियदायतानां तु ॥

Ashtoupancha cha pangthirvisthaare dairghyamapyameeshaam yath.
ashtirdvaadasa manavo ganitham kiyadaayathaanaam thu

Breadths of three rectangles are 8, 5 and 10 units and their lengths are 16, 12 and 14 units respectively What are the areas of rectangles?

Bhaskaracharya has explained the area of rectangle as the product of adjacent sides, while commenting the same rule given by Aryabhatta, in his book. Aryabhatta I has said that the product of length and breadth of a rectangle is its area.

Trapezium : Determination of area of a trapezium is more complicated than the above two classes of geometrical figures. Aryabhatta I has given the equation for the area of trapezium. Using this descriptions one can easily derive the modern equation. Aryabhatta has gone through a different route, which is intersting for a student of the subject (Aryabhateeya 2-8)

आयामगुणे पार्श्वे तद्योग हृते स्वपातरेखे ते ।
विस्तारयोगार्धगुणे ज्ञेयं क्षेत्रफलमायामे ॥

Aayaamagune paarsve thadyoga hruthe svapaatharekhe the
visthaara yogaardhagune jneyam kshethraphalamaayaame

Multiply the base and face of height, and divide (each product) by the sum of the base and the face. Results are

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prependiculars on the base and face. The result obtained by multiplying with half the sum of base and face by the height is to be known as the area (of trapezium).

The modern equation for finding out the area of the trapezium is the same. i.e $\frac{1}{2}(a+b)p$ where a and b are the opposite parallel sides and p the distance between them.

Bhaskaracharya gives an exercise for finding out the area using the equation in Bhaskarabbhashyam (63.1)

भूमिश्चतुर्दशस्यात् वदनं चैवरूपाणि ।
कर्णौ त्रयोदशाग्रौ संपाताग्रं फलं च वद ॥

Bhoomischathurdasasyath vadanam chaivaroopaani
karno thrayodasaagrow sampathaagram phalam cha vada

If the base of trapezium is 14 units, the face 4 units and the lateral sides 13 units each give out the junction line and area.

Quadrilateral: In Patiganita (rule 117) a method for determining the area of a quadrilateral is given exactly in the same way as that given by Brahmagupta for traingle. In fact, it is said that Brahmagupta has given the equation for quadrilateral which can also be applied to scalene traingles, when one side is considered equal to 0.

भुजयुतिदलं चतुर्धभुजहीनं तद्वधात्पदं गणितम् ।
सदशासमलम्बानामसदृशलम्बे विषमबाहौ ॥

Bhujayuthidalam chathurtha bhujaheenam thadvadhaathpadam ganitham
sadasaasamalambaanaama sadrusalambe vishamaabhow

(For finding out the area of a quadrilateral), set down half the sum of the (four sides of the quadrilateral) in four places, diminish them (respectively) by the sides, multiply, and take the square root. This gives the area of a quadrilateral.

This rule can be mathematically summarised as follows:

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$\sqrt{S \times (S-A)(S-B)(S-C)(S-D)}$; where A,B,C and D are sides and S half of the sum of sides (semi perimeter). This equation is also given in Brahmasputa siddanta XII 21(ii) and by Mahaviracharya in Ganitasara sangraha (VII 50 (ii) and also in Siddhanta sekhara (XII 28). In the above equation, if one side (Say D) is 0 then the quadrilateral becomes a triangle and the area will be $\sqrt{S \times (S-A)(S-B)(S-C)}$. This equation is also given, in another ancient mathematical book known as Yuktibhasha, for the explanation connected with cyclic figures.

When semi perimeter obtained by taking half of the sum of the sides of quadrilateral and the product obtained after each side is subtracted from the semi perimeter taking the square root of the product gives the area of triangle or quadrilateral. All the above equations/theorems in modern mathematics are known as Herons' equation/ theorem, even though they have been mentioned by Brahmagupta, Sreedharacharya, Mahaveera and Aryabhatta II. Brahmagupta had a sound knowledge in the cyclic figures too. He has given an equation for the diagonals of cyclic quadrilateral as follows "If one of the diagonals in a qudrilateral can be written as $\sqrt{(ab+cd)(ac-bd)/(ad+bc)}$ the other diagonal is $\sqrt{(ad+bc)(ac+bd)/(ab-cd)}$."

This is said to be one of the important contributions of Brahmagupta. Moreover, these two equations are meant for cyclic quadrilaterals, which are given with the equation for area. Hence there is another view among mathematicians that the equation for finding out the area of the quadrilateral given above by Brahmagupta; $\sqrt{S(S-A)(S-B)(S-C)(S-D)}$ is also for the cyclic quadrilateral and not for the non cyclic types. If so, Brahmagupta was correct too. The equation for the diagonals of the cyclic quadrilaterals is claimed to be the discovery by W. Snell [34] in 1919 AD, in Europe. Snell's claim was for an equation discovered more than one thousand years ago, by Brahmagupta, with all clarity of modern mathematics.

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Cyclic quadrilaterals: The mathematical equation for the radius of a circumcircle of a quadrilateral is given by Aryabhatta school as shown

R = 1/4 (ab+cd) (ac+bd)(ad+bd) ÷ (a+b+c-d) (b+c+d-a) (c+d+a-b) (d+a+b-c). No mathematican has given anything equivalent to this during that period. In the above equation the denominator (a+b+c-d)(b+c+d a)(c+d+a-b)(d+a+b-c) is written as (s-d) (s-b) (s-c) (s-a) where s is the perimeter equal to a+b+c+d. In this form the radius of the cyclic quadrilateral was rediscovered by Lhuiler in 1782 AD, centuries later then Indian discovery, put forward by the Aryabhatta school of mathematics.

In the 14th century, Narayana Bhatta developed two theorems connected with the cyclic quadrilaterals, which were not recorded earlier by any other foreign scientist. They are: 1. Three and only three diagonals are possible for four sides of cyclic quadrilaterals. 2. The area of cyclic quadrilateral is the product of these three diagonals divided by twice the circum-diameter.

The first theorem given above is also said to be Brahmagupta's contribution.

Different methods existed for finding out the area of quadrilaterals. Aryabhatta II has commented on determining different parameters of quadrilaterals without using the diagonals, thus given in Mahasiddhanta (XV. 70):

कर्णज्ञानेन विना चतुरश्रे लम्बकं फलं यद्धा वक्तुम्
वाञ्छति गणको योऽसौ मूर्खः पिशाचो वा ।।

Karnajnanena vina chathurasre lambakam phalam yaddhaa vakthum vaancchathi ganako yoasow moorkha: pisaacho vaa

A Mathematician who wished to tell the area or the altitude of a quadrilateral without knowing a diagonal is either a fool or

an insensible person.

In Brahmasphuta siddhanta (Ganitadhyaya XII.21) the method for the determination of the gross area of a triangle or a quadrilateral is given.

स्थूलफलं त्रिचतुर्भुज बाहुप्रति बाहुयोगदलघातः ।
भुजयोगार्धम् चतुष्टयोभुजोनघातात्पदे सूक्ष्मम् ॥

Sthoolaphalam thrichathurbhuja baahuprathi baahuyogadalaghaatha:
bhujayogaardham chathushtayo bhujonaghaathaalpadam sookshmam

The product of half of the sums of the opposite sides is the area of tringle or quadrilateral. I.e area = ½ (a+c) x ½ (b+d).

"Whenever a diagonal separates a quadrilateral into two halves, as triangles, the sum of the areas of the triangles will be the area of the quadrilateral": says Bhaskaracharya II in Lilavati.

Circles: Among many geometrical figures, circles and spheres have attracted the attention of both ancient and modern mathematicians, much more than any other figures. In circles the relations among the radius/diameter, circumference and the area are the three focussing points around which the studies were conducted. A series of theorems have also been developed during the 15th and the 16th centuries in Europe on the relations among these three parameters of circles. A detailed study on the subject could throw light on the ancient Indian contribution on this subject. It is a matter of fact that many of those theorems attributed to European mathematicians are really the contributions of Indians.

Circles have been dealt in detail in the Sulba sutras. To an extent of resonable accuracy, the relations among diameter, area and circumference have been given in these books. More interesting is the principle adopted for inscribing other geometrical figures in the circles. The growth of knowledge of

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ancient Indians on circles steadily increased to great depths and resulted in formulating theorems many of which are now known in the names of Newton, Kelvin, Gregory, Euler and others. The Indian contributions on these theorems/rules/equations took place centuries before the period of the Western Scientists.

Sulbasutras brought in application, the methods for constructing circular structures for the ritual altars. Aryabhatta (Aryabhateeya 2-13) defines drawing circles

वृत्तम् भ्रमेण साध्यम् वृत्तक्षेत्रं भ्रमेण साध्यतं ।

Vruttham bhramena saadhyam vruthakshethram bhramena saadhyath

Circles can be drawn by rotation-using a compass - This rule is also given in Brahmasphuta siddanta (XXII.7) and in Sishyadhivruddhi Tantra (2 VIII 2) by Lallacharya. Aryabhatta I in Aryabhateeya (2 10) gives the relation between the diameter and circumference of a circle accurately:

चतुरधिकं शतमष्टगुणं द्वाषष्टिस्तथा सहस्राणां
अयुतद्वय विष्कम्भस्यासन्नो वृत्तपरिणाहः ॥

Chathuradhikam sathatmashtagunam dvaashashtisthathaa sahasraanaam ayuthadvya vishkambasyaasnnon vruthbaparinaahu.

When 100 increased by 4 multiplied by 8 and added to 62,000 gives an approximate value for the circumference of a circle having diameter 20,000 units.

This gives a value of 62832/20,000 for ∏ and it is equal to 3 1416. This is the most accurate explanation for defining the value of ∏. Actual value of ∏ is 3.14285..... However Aryabhatta has approached this problem perfectly by saying that the value is near approximation using the Sanskrit word *asanno*. The accuracy level, indirectly followed till the first century BC and also in Sulbasutras was only upto 3.2. Bhaskaracharya I, while giving

his commentary to **Aryabhateeya** (60.1) gives a problem for determining the circumference of a circle from its diameter This shows that there were methods for calculating the unknown parameters using the known parameters of circles during the first half of the first millennia AD.

अष्टद्वादशषट्का: विष्कम्भस्तत्वतो मया दृष्टा: ।।
तेषां समवृत्तानां परिधिफलं मे पृथक् ब्रूहि ।।

Ashtadvaadasa shadkaa: vishkambhasthathvatho mayaa drushtaa:
theshaam samavrutthaanaam parithiphalam me pruthak broohi

Diameter of 3 circles are correctly seen by me to be 8, 12 and 6 units respectively. Tell me separately the circumference and areas of the circles.

This exercise, since written in the commentary of Aryabhateeya, is the application of the formula of 2∏r for determining the circumference of a circle having radius r. In the use of this formula, Aryabhatta's value of ∏ has been taken as 3.1416. A reverse method is also applied by Bhaskaracharya I in his usual style of giving the mathematical excercises, using fractions for finding out diameter from circumference. (Bhaskarabhashya 76.2)

नवनवयमरामाणामष्टाभि: शरयमांशहीनानां खखरसवृन्दस्य च मे
व्यासावचक्ष्व विगणस्य ।।

Navanavayamaraamaanaamashtaabhi: sarayamaamsaheenaanaam
khakharasavrundasya cha me vyaasaavachakshva viganasya

Calculate the diameter of a circle whose peripheries (circumference) are 3299 minus 8/25 units and 21600 units. methodology is application of reverse of the above formula. Another equation given by Virasena in his commentary called Dhavalateeka, written in 816 AD, on a mathematical work which was written much earlier by Pushpadatta namely Sakhandagama[65] says thus:

व्यासम् षोडशगुणितम् षोडशसहितम् त्रिरूपरूपैर्भक्तम् ।
व्यास त्रिगुणित सहितम् सूक्ष्मात् अपि तद् भवेत् सूक्ष्मम् ॥

Vyaasam shodasagunitham shodasasahitham thrirooparoopairbhaktham
vyasa thrigunitha sahitham sookshmaath api thath bhaveth sookshmam

The diameter multiplied by 16 increased by 16 divided by 113 and again combined with thrice the diameter is the circumference the more accurate than the accurate one. The above statement can be mathematically summarised as:

Circumference = 3d + (16d+16)/113

It is equivalent to Π =355/113+16/113d. The value is not correct (3 28328). But Hayashi, T Kusubha, T and Yano, M. have published the interesting information under the title: Indian value for Π derived from Aryabhatta in the Historica Scientifica[36] The significance of this work is not perhaps in its approximate value but in the application of round the way method, adopted Bhaskaracharya II in Lilavati (Kshetravyavahara-rule 40) gives relation between the circumference and the diameter, to arrive at the formula, as given by modern mathematicians.

व्यासे भवनन्दाग्नि हते विभक्ते खबाणसूर्यैः परिधिः ससूक्ष्मः ।
द्वाविंशतिघ्ने विहृतेऽथ शैलैः स्थूलोऽथवा स्याद् व्यवहारयोग्यः ॥

Vyaase bhava nandaagni hathe vibhakthe
khabhaana sooryai: paridhi: susookshma:
dwaavimsathighne vihruthe ऽtha sailai.
sthoolo ऽthavaa syaath vyavahaarayogya.

Multiply the diameter with 3927 and divide with 1250. It will give the correct circumference. Or multiply diameter with 22 and divide by 7, then the circumference for common purpose will be available.

First of the above answers is an approximate value for Π 3.1416 and the second is the accurate value 3.1428571....

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Charles M. Wish of the British East India Company has made this comment in the Royal Asiatic Society on some of the Ancient Indian achievements. "The approximation to the true value of the circumference with given diameters exhibited in these three works of Tanthrasangraha, Karanapaddhati and Sadratnamala, are so wonderfully correct, that European mathematicians who seek for such proportion in the doctrine of fluxions or in the more tedious continual bisection of an arc, will wonder by what means the Hindu has been able to extend the proportion to so great a length"[9]. These comments are on the three books written in the first half of this millennia whereas Aryabhatta I and Bhaskaracharya I have written in the middle of the first millennia i.e 499 AD and 628 AD. These two scholars have given accurately the values as 22/7 for arriving at the modern mathematical answer for Π. It is sure that scholars like Charles M. Wish would have got wonder struk if they had heard about these two mathematics/astronomy giants of the world who lived in India, 1500 years ago.

Theorems related with diameter and circumference of circles:

It is well known that the relation among the diameter, radius, circumference and areas of circle are as follows Circumference is equal to 3.14... x diameter or 2 x 3.14 x radius. Similarly the area of the circle is equal to 3.14.... x radius x radius. There are many theorems which give the relations among these parameters, which are important in modern mathematics. In Tantra sangraham (11) the following method has been given for finding out the circumference of the circle of known diameter. From this, the accurate value of Π can be derived.

व्यासे वारिधि निहते रूपहते व्यास सागराभिहते ।
स्त्रिशरादिविषम संख्या भक्तमृणं स्वं प्रथक् क्रमात् कुर्यात् ।।

Vyaase vaaridhi nihathe roopahathe vyaasa saagaraabhi hathe

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sthrisaraadvishama sankhya bhakthamrunam swam pruthak kramaath kuryaath

Multiply diameter with 4 and divide by 1 (first value). Again diameter is to be multiplied with 4 and divided separetely by odd numbers, 3, 5, 7, 9,. ... Subtract and add alternatively and separately to the first value to get the circumference correctly.

This theorem can be mathematically presented as follows:

Circumference = 4D/1 - 4D/3 + 4D/5 - 4D/7 + ... Where D is the diameter. Hence the value of ∏ = 4(1-1/3 + 1/5-1/7 + ...)

Earlier than the above author, Madhavacharya in Kriyakramakari (circle section II.40) (1350 AD) has given the same theorem and also adds at the end that if the addition and subtraction with odd numbers(alternatively) are done as many times as possible, more and more accurate value will be obtained for the circumference of the circle. (Indirectly, more accurate value for ∏ will be obtained)

Madhavacharya's non-terminating series for representing the circumference of a circle in terms of diameter, his invention of incommensurability of paridhisankhya has been placed on a very respectable position, as this theory is known to mathematicians in the name of James Gregory (1630-1675AD), who is said to have discovered it three centuries after Madhavacharya. A detailed explanation is given by Mukhopadhya and Adhikari in the historical development of the concept of ∏: a passage through India since 3000 BC [68]. This has been published in the Indian Science Crusader Journal. Some mathematicians have started calling this theorem, now, as Madhava Gregory theorem. A modified form of the above theorem is given in Karanapaddhati (6.1) by Puthumana Somayaji.

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व्यासाच्चतुर्घ्नात् बहुशः पृथक्स्थात् त्रिपञ्चसप्ताद्ययुगाहृतानि ।
व्यासे चतुर्घ्ने क्रमशस्त्वृणम् स्वं कुर्यात् तदा स्यात् परिधिः सुसूक्ष्मः ॥

Vyaasaacchathurghnaath bahusa: pruthaksthaath thripancha saptbaadyayugaahruthaani
vyaase chathurghne kramasasthrunam svam kuryaath thadaa syaath paridhi: susookshma:

Four times of the diameter is to be divided separately by each of the odd integers 4, 5, 7,...... Every quotient whose order is even, is taken away from one preceding it The combined result of all such small operations when subtrated from four times the diameter, gives the circumference with progressively greater accuracy.

i.e. Circumference = 4D-4D (1/3-1/5)-4D(1/7-1/9)....

Or Π/4=(1-1/3+1/5-1/7+.....)

A famous ancient anonymous commentary for this theorem explains it with example using katapayadi number system. Katapayadi numbers are given in parenthesis. This commentary appears in Karanapaddhathi published by the University of Kerala.

Say 113 unit (lakutam) is the diameter of a circle. Multiply it with 4 to get 452 (gramavil) divide by 3 and reduce it from gramavil to get 301 (pannagam) Repeat division of 452, by 5 and subtract from former to get 391. When repeating with 7 we get 326, continue with 9,11,13..... finally the answer will be 355 (sanmali) i.e the circle having diameter 113 (lakutam) will have a circumfernce 355 (sanmali). Hence the value of Π = 3.1415929... Modern value is 3.1428571.....

Another theorem given on this relation, follows a very complex mathematical procedure. There may not be anyone to put forth their claim in the world of mathematics, as the

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discoverer of these theorems, except Puthumana Somayaji himself, who is the author of Karanapaddhathi. (6.4).

वर्गैर्युजाम् वा द्विगुणैर्निरेकैर्वर्गीकृतैर्वर्जित युग्मवर्गैः ।
व्यासं च षड्घ्नम् विभजेत् फलं स्वं व्यासे त्रिनिघ्ने परिधिस्तदास्यात् ॥

Vargairyujaam vaa dvigunairnirekair vargeekruthair varji thayugma vargai. vyaasam cha chadghnam vibhajeth phalam svam vyaase thrinighne paridhi sthadaasyaath

Six times the diameter is divided separetely by the square of twice the square of even integers 2,4,6.... minus one, diminished by the squares of even integers themselves. The sum of the resulting quotient by thrice the diameter is the circumference.

This can be mathematically written as follows:Circumference =

$$3D+6D([1/2x2^2-1]^2-2^2) + ([1/2x4^2-1]^2-4^2)+[(1/2x6^2-1)^2-6^2]) + \ldots$$

It is obvious that to derive a formula at such a level of sophistication an unparallel capability in mathematics is required. Perhaps that may be the reason behind the remark made by Charles M. Wish on the authors of the three books including Karanapaddhathi (given earlier)

Another theorem to calculate the circumference from diameter is also given in Karanapaddhathi (6.2). There may not be any claimant for this theorem also.

व्यासाद् घनसंगुणितात् पृथगाप्तं त्र्याद्ययुग्विमूलघनैः ।
त्रिगुणव्यासे स्वमृणम् क्रमशः कृत्वापि परिधिरानेयुः ॥

Vyaasaath vanasangunithaath pruthagaaptam thryaadyayugvimoolaghanai:
thrigunavygaase svamrunam kramasa. kruthvaapi paridhiraaneyu:

Multiply the diameter of a circle with 4 and keep it at different places and divide each with the odd numbers beginning

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from 3, 5, 7,... as their cubes subtracted by the same value. Repeat this and add/subtract alternatively the results to three times the diameter of the circle to get the circumference with the highest degree of accuracy. This theorem can be mathematically represented as follows:

$$\text{Circumference} = 3D + 4D/(3^3-3) - 4D/(5^3-5) + 4D/(7^3-7) - \ldots$$

The anonymous commentary given in the Karanapaddhathi, for this theorem gives the worked out example. If the diameter is 113, multiplied with 4 gives, 452, cube of 3 reduced by 3 gives 24 added this to 3 x 113 gives 538 , 5 reduced from the cube of the 5 gives 120 and when 452 is divided by 120 and that reduced from 358 gives 354 and repeat with 7,9, et., finally reach 355.

Karanapaddhathi (6.6) gives a rule for finding out the circumference or diameter of a circle when one of these parameters and both from another reference circle are known.

गुणहारकभूतैस्तैर्व्यासवृत्तैर्यदोदितम् ।
इष्टवृत्तान्नयेद् व्यासं व्यासाद् वृत्तम् विपर्ययात् ॥

Gunahaarakabhoothaisthairvyaasavrutthairyadoditham
ishtavrutthaannayeth vyaasam vyasaad vruttham viparyayaath

Take the known measurements of the circumference and respective diameter of a circle. From the ratio of circumference to diameter the circumference of an unknown circle can be determined, when its diameter is known. Similarly from the circumference, the diameter can also be deduced.

This is an indirect method of calculating the value of circumference/ diameter (∏). C/D x Di = Ci and D/C x Ci = Di where C and D are known parameters of a circle and Ci and Di, one of which is known at a time.

Determination of hypotenuse for a Rt triangle principle

was used by Aryabhatta for finding out the radius of a shadow circle formed by the light on a lamp post on a gnomon. Shadow circle is formed from the top tip of the gnomon as the centre and the radius(from the tip) end touching the end of shadow. According to Pythagorus theorem, square root of the sum of squares of height of gnomon and length of the shadow gives hypotenuse and the same is the radius of shadow circle. (Aryabhateeya 2.14)

शङ्को: प्रमाणवर्गम् छायावर्गेण संयुतं कृत्वा ।
यत्तस्य वर्गमूलं विष्कम्भार्धं स्ववृत्तस्य ।।

Sanko: pramaanavargam cchayaavargena samyutham kruthvaa
yatthasya vargamoolam vishkambhaardham svavrutthasya

Add the square of the height of gnomon to the square of its shadow. The square root of that sum is the semi diameter of the circle of shadow. (this problem is also given in Khandakhadyaka (I.iii.10) and in Mahabhaskareeya (III.4):

नृच्छायाकृतियोगस्य मूलमाहुर्मनीषिण: ।
विष्कम्भार्धं स्ववृत्तस्य छाया कर्मणि सर्वदा ।।

Nrucchaayaakruthiyogasya moolamaahurmaneeshina:
vishkambhaardham svavrutthasya cchayaa karmani sarvadaa

The same explanation can be given for this line also. i.e The sum of squares of the height and the distance of the shadow gives the distance from the top tip of the gnomon to last end of the shadow.

Angular dimensions: There are two types of dimensions used in measurements. The linear and angular measurements. All the common measurements are in the linear dimensions which ranges from 0 to infinity measurements related to the circles, spheres, etc., have angular measurements ranging from 0 to 360°. Each

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degree angle in mathematical Sanskrit is called amsa or bhaaga. In the angular measurements a circle has 360° measurements. In katapayadi number system, a circle had 21600 (360 x 60) kala (minutes) mentioned as *a na tha pu ram* minutes (i e 21600 minutes) Each degree is further divided into 60 minutes, called kala and each minute(′) into further seconds (″) which is called vikala Further down each vikala subdivided into sixty parts known, as thalpara and further down to prathalpara. Thalpara and prathalpara are also known as liptam and praliptam. It is commonly believed that all these angular values are the contributions of modern science The knowledge on the angular dimensions, as it is known now in exactly the same way, is Indian contribution. To cite an example, given here is a quotation from Vateswara siddhanta II i. 49 (c-d). 50 on the angular values given in Bhootha sankya system:

अंगगुणवेदभूताशाः कलिका विकलाः समुद्रजलधयः
स्वल्पजलखाष्टशशि धृतिशशिनः कलिकाः शरागनयो विकलाः
त्रिण्याकृतिरष्टनवत्रिभुवो विश्वे जिनांशाप्त्या ।।

Angagunavedabuthaasaa: kalikaa vikalaa: samudrajaladhaya:
swalpajalakhaashtasasi dhruthisasina: kalikaa: saraagnayo vikalaa:
thrijyaakruthiwarashta navathribhuvo visve jinaamsajyaa.

3437′ 44″ is radius and 11818047′ 35″ is square of radius and 1398′ 13″ (both are values in radians) is Rsin of 24°.

Angular dimensions in minutes and seconds are given in the above figures. These values are arrived at by the method of dividing angle of a circle 360° by 2 x 3 14 (because 2Πr = 360° i.e r = 360°/2 x Π, when multiplied with 60 for converting into minutes - 3436′ 21″ is obtained. The modern answer for radius in angular measurements. Vateswara could give the above value 3437′ 44″ to a great level of accuracy, about 1300 years ago.

Later Puthumana Somayaji has also followed the same line.

The angular dimension has been given with a greater level of accuracy. The value in Karanapaddhathi (6.7) is given in katapayadi system.

अनूननूनाननूनुन्नित्यैस्समाहताश्चक्रकलाविभक्ता: ।
चण्डांशुचन्द्राधमकुम्भिपालैर्व्यासस्तदर्द्धं त्रिभमौर्विकं स्यात् ॥

Anoonanooothnaananooonoonnanithryai ssamaahathaaschakra kalaavibhakthaa: chandamsuchandraadhamakumbhi paalai rvyaasasthadarddham thribha mourvikasyaath

When the diameter is 10,000000000, the circumference is 31415926536. From the angle in radians can be obtained.

3437′ 44″ 48‴ 22′′′′ is the angular radius of a circle and its square is 11818102′ 50″ 40‴ 3′′′′ 15′′′′′ 20′′′′′′ 4′′′′′′′)

Upto this level no mathematician has attempted to find out the value for angular measurements.

Area of circles: Area of circle is the product of 3.14 and the square of its radius. This has been directly given by Aryabhatta. (Aryabhateeya 2.7)

समपरिणाहस्यार्धं विष्कम्भार्धं हतमेव वृत्तफलं ॥

Samaparinaahasyaardham vishkambhaardha hathameva vrutthaphalam

Half of the circumference when multiplied with half of the diameter gives the area of the circle (i.e Half of the circumference is 2 x 3.14 x d/2, this when multiplied with r, the area is obtained 3 14 x r x r i.e Πr^2)

According to Brahmagupta the area of the circle (Sanskrit lines quoted in Lilavati along with the area of sphere are given elsewhere) is "The product of circumference and one fourth of the diameter gives the area of a circle. Four times of this area gives the area of sphere (having the radius of the circle)".

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This can be mathematically represented as: 2 x 3.14 x r x 1/4 x diameter. This is the area of circle. I.e 3.14 x r x r. When this is further multiplied with 4, it becomes: 4 x 3.14 x r x r, which is the area of the sphere.

The same formula is said to be given in Tatwarthadigama sutra bhashya (III.11) of Umavati, a manuscript belonging to the first century AD and also in Bruhatkshetra samasa (1 7) of Jinabhadragani (609 AD). It shows that from the very early days determining the area of a circle to a very high level of accuracy was known. Bhaskaracharya I, in his commentary has given this problem for determining the area of a circle (75.1)

द्विचतुस्सप्ताष्टानां व्यासानां यानि वृत्तगणितानि ।
सूक्ष्मासन्नानि सखे विगणय्य गणितानुसारेण ॥

Dwichathu saptaashtaanaam vyasaanaam yaani vruthaganithaani
sookshmaasannani sakhe viganaya ganithaanusaarena

O! friend calculate according to the ganita of Bhatta (Aryabhateeya) the nearest approximate of the areas of the circles whose diameters are 2, 4, 7, and 8 units.

In this problem from the given radius, the area of a circle can be determined. Bhaskaracharya II, in Lilavati, has given another method for determination of area, approximately and also accurately, of a circle form its diameter (page 254 rule 42)

व्यासस्य वर्गे भनवाग्निनिघ्ने सूक्ष्मं फलं पञ्चसहस्रभक्ते ।
रुद्राहते शक्रहतेऽथवा स्यात् स्थूलं फलं तद्व्यवहारयोग्यं ॥

Vyaasasya varga bhanavaagninighne sookshmam phalam
panchasahasra bhakthe rudraahathe sakrahatheathavaa
syaath sthoolam phalam thadvyavaharayogyam

When the square of the diameter multiplied with 3927 and divided by 5000 the accurate area of circle will be obtained. But

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when the square of diameter multiplied with 11 and divided by 14, approximate value will be obtaintained.

However, the correct answer happened to be given in reverse. By following the former method approximate area of a circle, which is equal to .7854 x d x d, will be obtained. By the second method accurate value of the area of a circle equal to .7857142 x d x d will be obtained.

Polygonal approximation : A complicated mathematical procedure reported recently, after detailed study and investigation, is one the polygonal approximation to circles given by Madhavacharya, six centuries ago. This appears to be a remarkable contribution of the great Indian mathematician. This has been derived carefully by Mukhopadhyaya and Adhikari. [69]

चतुर्भुजे दोः कृतिनाग भागमूले हरो हारभुजांघ्रिभेदात्
भुजाहताद्धारहृतं तु कोणान्नीत्वा विलिख्याष्टभुजाः प्रसाध्याः
अष्टाश्रदोर्धकृतिर्निधेया व्यासार्धवर्गे पदमत्र कर्णे
तेनाऽऽहरेद् दोर्दलवर्गहीने व्यासार्धवर्गं यदतः फलं स्यात्
तदूनकर्णो दलितो हराख्यो गुणस्तु विष्कम्भदलोनकर्णः
भुजार्धमेतेन हते गुणेन हरेण भङ्क्त्वा यदिहापि लब्धम्
तत्कोणतः पार्श्वयुगेषु नीत्वा छिन्नेऽन्तरे स्यादिह षोडशाश्रम्
अनेन मार्गेण भवेदतम्भ्व रदाश्वर्क वृत्समतश्च साध्यम्

Chathurbbuje do: kruthinaaga bhaagamoole
haro haara bhujaamghri bhedaath
bhujaahathaadvaarahrutham thu konaanneethvaa
vilikhyaashtabhugaa: prasaadhyaa:
ashtaasradorardhakruthirnidheyaa
vyaasaardha varge padamathra karne
thenaaahared dordala vargaheenam
vyaasaaardhavargam yadatha: phalam syaath

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thadoona karno dalitho haraakhyo
gunasthu vishkambadalona karna:
bhujaardhamethena hatham gunena
harena bhangthva yadithaapi labdham
thathkonatha: paarsva yugethu neethvaa
cchineeafnthare syaadiha shodasaasram
anena maargena bhavedathascha
radaasvarka bruthsamathascha saadhyam

The square root of one eighth part of the square side of quadrilateral is hara. Haraka less 1/4 of the side is multiplied by the side and (the product) divided by hara. Taking segments equal to the result thus obtained along a side from its corners (i e taking segments equal in length to this result on every side from this corners on it) the octagon is formed

Adding the square of half of a side of the octagon to the square of radius, the square root of the sum obtained is (called) the diagonal By that diagonal is divided the square of the radius less half the square of a side (of octogon). The result is subtracted from the diagonal and (the difference) is halved to get what is called hara. The diagonal less held the diameter is multiplied by half of a side (of the octagon), the product (on being) divided by hara gives, which is taken (for length of segment) along sides (measured) from corners (i e on every side from either of it) and out of to get a (regular) polygon of 16 sides. In this way of cutting portions of each side (in between the corners on it) ultimately a circle can be obtained.

This is a method to derive a circular figure from polygons having many sides. It took the modern science many years to understand Madhavacharya's above noted procedure.

Sankara has given an explanation for the above procedure. First, start square of 1/8 in part of square of a side = $\sqrt{x^2/8}$ as hara. Haraka less 1/4 th of a side $(\sqrt{^2/x/8})-x/4$ as kou karananatra.

This multiplied by a side and (the product) divided by hara or haraka get circumference when x is the diameter as

$$\text{circumference} = x((\sqrt{x^2/8})-x/4)/\sqrt{x^2/8}$$

Polygonal approximation to circles involves construction of regular polygons of sides $n (n \geq 4)$ either circumscribing a circle or inscribing it. The explanation given here is the reproduced version from the Journal, to show that this level of complexity on the mathematical derivation could be achived by the ancient Indians.

Area of Spheres : Spherical body is an extension of circles. Circle and spheres are related in different ways. From the area of the circles, the surface area of the sphere can be calculated. Indian mathematicians have calculated area of planet earth, using the formula for calculating the area of sphere, obtained from the diameter of earth (1050 yojanas). Brahmagupta's quotation for determining the area of sphere is given. It has been mentioned that the product of 1/4 of the diameter and circumference will give the area of a circle. This when multiplied with 4 gives the area of sphere.

Aryabhatta II in Mahasiddanta (XVI) has also shown circumference of a sphere, when multiplied with its diameter gives the area of the sphere $2\Pi r \times 2r = 4\Pi r^2$. In Lilavati (p 281 - 41) the following stanza occurs.

वृत्तक्षेत्रे परिधिगुणितव्यासपाद: फलं तत्
क्षुण्णं वेदैरुपरिपरित: कन्दुकस्येव जालं ।
गोळस्यैवं तदपि च फलं पृष्टजं व्यासनिघ्नम्
षड्भिर्भक्तम् भवति नियतम् गोळगर्भे घनाख्यम् ।।

Vrutthakshethre paridhigunitha vyaasapaada: phalam thath
kshunnam vedairupari paritha kandukasyeva jaalam
golasyaivam thadapi cha phalam prushtajam vyaasanighnam
shadbhirbhaktham bhavathi niyatham golagarbhe ghanaakhyam

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When circumference is multiplied with diameter and that result divided by 4, that will give the area of a circle. This when multiplied with 4 gives the surface area of the globe which is like surface of a ball. This when multiplied with diameter and divided by 6 gives the volume of the sphere of globe.

Mathematically it can be written as $2\Pi r \times 2r/4 = \Pi r^2$

This is for the area of the circle. This when multiplied with 4 the area of the sphere will be obtained. The next part of the above stanza deals with the volume of the sphere.

In all the above equations, the area of a circle and that of a globe have been given correctly. Hence, it can be proved that, the Sanskrit literature in mathematics carried equations for the area of circles, spheres, and related structures. Their contributions exactly conform to modern mathematical knowledge.

Arcs and chords of circles: It is interesting to note that the arcs, part of the circumference and chords, the line joining the two ends of the arcs, were very well known to Indian mathematicians. Aryabhatta I gives the rule which is an important common geometrical theorem, on the relation between the chord and the radius of a circle. (Aryabhateeya 2 9)

परिधेः षड्भागज्या विष्कम्भार्धेन सा तुल्या ।

Paridhe: shadbhaagajyaa vishkambhaardhena saa thulyaa

The chord of one sixth of circumference is equal to the radius of that circle. This theorem appears to have been known still earlier, since it is given in one of the Siddhantas included in Panchasiddhantika of Varahamihira (IV 2 a,b). The relation between chord and radius of circle is also given by Puthumana Somayaji in Karanapaddhathi (6-19)

स्वस्वचापघनषष्ठभागतो विस्तरार्द्ध कृतिर्भक्तवर्जितम् ।

शिष्टचापमिहशिञ्जनी भवेत् तद्युतोऽल्पकगुणोऽसकृत्धनुः ॥

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Svalpachaapaacchaghanashashta bhaagatho vistaraardhakruthir-
bhaktha varjitham sishtachaapamthasinjanee bhaveth
thadyutho falpakaguno fasakruthdhanu:

The chord of an arc of a circle is obtained from the result of the cube of the length of the arc divided by six times the cube of radius and subtracted from the arc. This can be mathematically presented as follows:

Chord (R Sine θ) = $s - (s^3 / 6r^2)$. Here length of the arc s is in angular dimensions, r is the radius and θ is the angle of the arc.

Another relation between circumference and the arc is given by Bhaskaracharya II in Lilavati (p. 300. rule 49)

व्यासाब्धिघातयुतमौर्विकया विभक्तो जीवाङ्घ्रिपञ्चगुणित: परिधेस्तुवर्ग: ।
लब्धोनितात् परिधिवर्ग चतुर्थभागादाप्ते पदे वृतिदलात् पतितेधनु: स्यात् ।।

Vyaasaabdhighaathayuthamourvikayaa vibhaktho
jeevangghri panchagunitha: paridhesthuvarga:
labdhonithaath paridhivarga chathurtha
bhaagaadaapte pade vruthidalaath pathithedhanu: syaath.

One fourth of five times the chord multiplied with square of circumference divided by four times the diameter added with the chord. This value is subtracted from one fourth of the square of circumference. Square root of this is taken and subtracted from half of the circumference to get the arc.

I.e Arc = ½ circumference- $\sqrt{}$(1/4 circumference^2) - (1/4 (chord x 5x circumference) + 4 x (diameter + chord)).

In Lilavati (rule 48 page 298), the following relation of the diameter and the circumference with the chord is also given by Bhaskaracharya II.

चापोननिघ्न परिधि: प्रथमाह्वय: स्यात् पञ्चाहत: परिधिवर्गचतुर्थभाग: ।
आद्यो नितेन खलु तेन भेजच्चतुर्घ्न व्यासहतं प्रथममाप्तमिह ज्यका स्यात् ।।

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Chaaponanighna paridhi: prathamaahvaya:
syaath panchaahatha: paridhi varga chathurtha bhaaga:
aadhyo nithena khalu thena bhejacchathurghna
vyaasaahatham prathamamaapthamiha jyakaa syath

Arc multiplied with the circumference minus arc is termed as first result. The first result is subtrated from one fourth of five times the square of circumference give the second result. The first result multiplied with 4 times the diameter and divided by the second result, gives the chord.

I.e. Chord = 4 x diameter (Circumference - arc) arc ÷ 5/4 x Circumference^2- (circumference - arc) arc.

Lilavati 43b and 44b gives another interesting equation:

ज्याव्यासयोगान्तरघातमूले व्यासस्तदूनो दलितः शरः स्यात् ।
व्यासाच्छरोनाच्छरसंगुणा च मूलं द्विनिघ्नम् भवतीह जीवा ।
जीवार्धवर्गे शरभक्तयुक्ते व्यासप्रमाणं प्रवदन्ति वृत्ते ।

Jyaavyaasayogaanthara ghaathamoolam vyaasasthadoono dalitha:
sara: syaath vyaasaaccharonaacchara sangunaa cha moolam
dvinighnam bhavatheeha geevaa yeevaardhavarge sarabhaktha
yukthe vyaasapramaanam pravadanthi vrutthe

When the sum and differences of diameter and the chord are multiplied, and their square root is taken and if half of that is subtracted from the diameter, the arrow is obtained. The difference of diameter and the arrow multiplied with the arrow, twice the square root of that value gives the chord. The square of half the chord divided by arrow and added with arrow gives the diameter of the circle.

This rule is found to be correct and gives an example of the inner vision of the ancient mathematicians to look into various

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aspects of the inscribed geometrical figures Aryabhatta in Aryabhateeya (2-17) had also given, yet another relation between the diameter and the chord.

यश्चैव भुजावर्ग कोटिवर्गश्च कर्णवर्गः सः ।
वृत्तेशरसंवर्गो/र्धज्यावर्गः य खलु धनुषोः ।।

Yaschaiva bhujaavarga kotivargascha karnavarga: sa:
vruthho sara samvargo fardhajyaavarga: ya khalu dhanusho:

In a Rt. (one angle 90°) triangle, square of the base plus square of the upright is the square of hypotenuse. In a circle (when chord divides it into two arcs) the product of the arrows of two arcs is certainly equal to the square of half the chord.

The second part of the theorem states that, in a circle if a chord CD and diameter AB intersect each other at right angles at E then: $AE \times EB = CE^2$. The first part of the theorem (similar to Pythagorus theorem) when applied for a right angled traingle inscribed in a circle is known as Hankel's theorem also known as the theorem of hypotenuse. This theorem has been in Aryabhateeya, centuries before Hankel. Hence Hankel's theorem rightly belongs to Aryabhatta I.

Angular relation with arcs & chords in circles : It is obvious that the relation of an arc with the circumference or diameter can only be through the angular measurements of arc. Higher the values of angles the longer will be the arc and also the chord, in a circle, upto a limit. Similarly, for the same angle, the higher the diameter the higher will be the length of the arc. In fact dozens of equations and theorems, known by the names of European mathematicians have been put forth and applied in modern mathematics. Many of these theorems belong to Indians. A few among these theorems are quoted here.

Sine, Cosine and Tangent of angles: In modern mathematics the use of sine, cosine and tangent of angles are very common. In

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a right angled triangle if θ is considered one of the angles, the sin θ of that angle is equal to the ratio of length of the opposite side and hypotenuse. Cos θ is the ratio of adjacent side and hypotenuse. Tan θ is the ratio of the opposite side and adjacent side. It is believed that these three are the contributions of the Western or European mathematicians. In ancient India these were well known even during the period of Aryabhatta R sine θ is commonly referred as a Bhuja or Bhujajya or Jya. R cos θ is referred as the Kotijya or Kojya or koti Even though tan of the angles was used in the theorem, there is no proof on the use of the specific term for tan θ. In the applied mathematics sin θ/ cos θ was used in India from very olden days. Knowledge of the most complicated branch of mathematics i e sine, cosine and tangent undoubtedly had their origin in India. One of the achivements of ancient Indians is the R sine θ values given by Aryabhatta I in Aryabhateeya (1-12) in 499 AD. It is given in the Aryabhateeya number system. Values for the Rsines of angles at interval of 225′ of the angular values are given. I.e Rsin 225′, 450′, 675′. . etc., It is presented as such and the corresponding modern values are also given. This is similar to the modern Clarke's table. Aryabhatta's table for R sin θ and R sine difference are given for 0° to 90° angular values at intervals 3° 45′. Many mathematicians have made use of this table for calculating R sine values.

मखि भखि फखि धखि णखि ञखि ङखि हस्झ
स्ककि किष्ग श्घकि किघ्व ।
घ्लकि किग्र हक्य धकि किच्य स्ग श्झ ङ्व
क्ल प्त फ छ कलार्धज्या ।।

Makhi bhakhi phakhi ghakhi nakhi njakhi ngakhi hashja skaki
kishga sdhaki kidhva dhlaki kigra hakya dhaki kichya sga
· shha ngva kla ptha pha ccha kalaardhajyaa

225 (224′.856),	224 (223′.893)	222 (221′.971),	219 (219′.100),
215 (215′.289),	210 (210′.557)	205 (204′.923),	199 (198′.411),
191 (191′.050),	183 (182′.872),	174 (173′.909),	164 (164′.033),
154 (153′.792),	143(142′.724),	131 (131′.043),	119 (118′.803),
106 (106′.053),	93 (92′.850),	79′ (79′.248),	65 (65′.307),
51 (51′.087),	37 (36′.648),	22 (22′.051),	7 (7′.361)

These are R sine differences (at the intervals of 225 minutes of the angle of an arc) In the parenthesis are given the modern values based on Clarke's table. The R sin for 3600′ (2978′) was modified by Aryabhatta II and corrected as 2977′ instead of 2978, the value is wonderfully very near to modern value of 2977′.395. The tables given in Suryasiddhanta and others like Bhaskara II and Aryabhatta II, in their respective books are the same.

Using the above table other mathematicians have described the method for the determination of length of arcs and also the method for finding out the Rsin values of the angles falling in between 225′ angles. Bhaskaracharya I in Mahabhaskareeya (IV-3-4 (I) gives this explanation:

लिप्तीकृत्य हरेनमख्या जीवा लब्धस्ततः पुनः
वर्तमानाहतं शेषं मख्याचैव विभाजयेत् ।
पूर्वसङ्कलिते युक्ते ज्याक्रमेणोत्क्रमेण वा
स्वपरिध्याहतेऽशीत्या लब्धं क्षयधनं फलम् ॥

Lipteekruthya harenamakhyaa jeevaa labdhasthatha: puna:
varthamaanahatham sesham makhyaachaiva vibhaajayeth
poorva sankalitheyukthe jyaathkramenothkramena vaa
svaparidhyaahatheaseethyaa labdham kshayadhanam phalam.

Reduce the arc (from the degree angular values) to minutes and then divide by 225, the quotient denotes the number of R

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sin differences to be taken completely (taking the Rsine 225′ from 0 - 90°). Then multiply the remainder by the next Rsine difference and divide by 225. Add the quotient to the sum of the R sine difference obtained before. The sum thus obtained is the R sine of the arc.

This procedure is exactly the same as that we follow in using Clarke's table. In the above rule mentioned by Bhaskaracharya I, Aryabhatta's table is used as reference and other values are derived from it. This method is also given in Suryasiddhanta (2, 31-32) and Brahmasphuta siddhanta (2.10). The application of the method and its scientific reliability for calculating Rsine of 32° is given here as an example (Mahabhaskareeya).

Reduce 32° into minutes which is equal to 1920′, divided by 225 to get 8 as quotient and the remainder is 120′. The sum of 8 Rsines from the Aryabhatta's table is equal to 1719′. Multiply the remainder 120 with 9th Rsine (as given in Aryabhatta's table i.e 191′) and dividing the product by 225 we get 101′ 52″. Adding this to the previous sum (1719), we get 1820′ 52″. According to Bhaskara's method R sin of 1920′ is 1820′ 52″ whereas the modern value for R sin 32″ is 1821′ 43″. Remarkably nearer!

Brahmagupta has arrived at this equation for finding out the value for any angle θ less than 225 for 't' times of 225', where 't' be an integral number. Then 225' t + θ' = Sum of 't' R sine difference + θ'/225 (t th Rsine difference + (t+1) the Rsine difference) x ½ - θ'/225 (t th R sine difference - (t+1)th Rsine difference) x ½.

This formula is also given in Khanda khadyaka (9.8) and in Siddhanta siromany of Bhaskaracharya II (I,ii 16). In Parameswara's commentary for Laughubhaskareeya (II.2 (ii)-3(I) a similar explanation is given. Another method for determining the Rsine is given by Nilakanta in Tantra sangraha (II 10-13),

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which can be mathematically presented as follows.

$R \sin(225't + \theta') =$ (Sum of t Rsine differences $+ \theta \times$ (R cos (225'(t+1)) + T cos (225 t)/2R.

Khandakahadhyaka (I X - 14) gives a rule for finding out the arc of an angle, when its sine value is known, by using the method interaction. The basic text from which the method was derived is given here. It appears from the Sanskrit pandit's opinion that the translation of the Khanda khadhyaka text was tough to a mathematician. However the standard translation by Prof. P.C. Sengupta[70] is given here.

This translation has wide acceptability and true to its content.

चापानयने नवशतविकलवधाद् भोग्यलब्धलिप्ताभिः
कृत्वा खण्डक्रमसकृत् तल्लब्धकला विकलचापम् ॥

Chapaanyane navasathavikalavadhaad bhogyalabdhalipthaabhi:
krutvaa ghandakrama sakruth thallabdha kalaa vikalachaapam

In finding the arc corresponding to a given sine, find the residue left after subtracting as many as possible of the tabular differences of sines, multiply it by 900 and divide by the tabular difference to be passed over; by means of the minutes of arc obtained. Find the true tabular difference by repeating the process and thus finding the minutes of arc corresponding to the required, to reduce the sine.

Devacharya, in Karanaratna (I.23) has given the R sin θ for the angles θ at intervals of 10° for the value of R (300')

रामोनु रत्नाद्य नृमान्य लुब्धको नागग्र निस्तार खजाग्र माधुरः ।
ज्ञानांगमित्यत्र नव प्रकीर्तिता जीवाह्वनन्ताप्तफलैः समन्विताः ॥

Ramonu rathnaddya nrumaanya lubdhako
naagagra nisthaara khagaagra maadhuraa

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gnaanaangamithyathra nava prakeerthithaa
geevaabhavananthaapthaphalai samanvithaa:

(If R,- the angular radius- is 300') 52, 102, 150, 193, 230, 260, 282, 292 and 300 are the Rsine values at intervals of 10° To obtain the R sine of the arc, divide it into smaller arcs of 600' each and add the Rsine difference corresponding to them. This shows that similar tables to that of Aryabhatta I have been prepared by Indian mathematicians for the sine values of angles.

Thus it can conclusively be said that the systematic method of discovering and applying the sine value is the contribution of Aryabhatta I in 499 AD. Even though Aryabhatta himself has not given the method followed, Bhaskara I gives it in detail. However Aryabhatta gives the explanation in short form (Aryabhateeya 2-11)

समवृत्तपरिधिपादं छिन्द्यात् त्रिभुजाच्चतुर्भुजाच्चैव ।
समचापज्याधानि तु विष्कम्भार्धे यदेष्टानि ।।

Samavruttha pardhi paadam chhindyaath thribhujaa chhathurbhujaachhaiva
samachaapajyaardhani thu vishkambhaardhe yadeshtaani

Divide a quadrant of the circumference of a circle into as many parts as desired. Then from the Rt triangles and quadrilaterals one can find as many Rsines of equal arcs as one likes for any given radius.

Accordingly the values Aryabhatta derived is given here as exapmles.

Rsin 30°	=	$R/2$	= 1719'
Rsin 60°	=	$\sqrt{3}/2R$	= 2978'
Rsin 15°	=	$\frac{1}{2}\sqrt{(R\sin 30^\circ)^2 + (R \text{ verse } 30^\circ)^2}$	= 890'
Rsin 75°	=	$\sqrt{R^2 - (R\sin 15^\circ)^2}$	= 3321'
Rsin 45°	=	$R/\sqrt{2}$	= 2431'

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Given above are the exact values according to modern mathematics. This was based on the value of R in radians (in minutes) as 3438′.

Bhaskara in Mahabhaskareeya (VII, 17-19) has given the method for determining the Rsine of acute angles (angles which are less than 90°)

मख्यादिरहितं कर्म वक्ष्यते तत्समासतः चक्रार्धांशक समूहाद्विशोध्या
ये भुजांशकाः तच्छेषगुणिता द्विष्ठाः शोध्याः खाभ्रेषुखाब्धितः
चतुर्थांशेन शेषस्य द्विष्ठमन्त्यफलं हतम् बाहुकोट्योः फलं कृत्स्नम्
क्रमोत्क्रमगुणस्य वा लभ्यते चन्द्रतीक्ष्णांश्वोस्ताराणां वापि तत्त्वतः

Makhyaadirahitham karma vakshyathe thathsamaasatha:
chakraardhamsaka samoohadvisodhyaa ye bhujaamsakaa:
thacchesbagunithaa dvishtaa: sodhyaa: khaabhreshukhabdhitha:
chathurthaamsena seshasya drishtamanthyaphalam hatham
bhaahukotyo: phalam kruthsnam kramolkrama gunasya vaa
labhyathe chandratheekshnaamsvosthaaraanaam vaapi thathvatha:

Subtract degrees of Bhuja (or Koti) from the degrees of half of a circle (180°). Then multiply the remainder by the degrees of Bhuja (or koti) and put down the result at two places. At one place subtract result from 40500. By one fourth of the remainder (obtained), divide the result at the other place as multiplied by the antyaphala. Thus obtained is the bhahu phala (or Kotijya) for the Sun, moon, stars or planets.

This theorem was given for astronomical calculations and can be suitably adopted for mathematical calculations too:

$$R \sin \theta = 4\theta\,(180\,\theta - \theta)\,R/(40500 - \theta\,(180 - \theta)$$

This theory has been proved by Prof. K. S. Sukla [71], and detailed discussion is given in the book published by the Department of Mathematics and Astronomy, Lucknow

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University. On the above equation Sukla states that "the Indian mathematicians were aware by latest 9th century AD that an infinite convergent series has a finite sum as given above".

Values of Rsine θ given by ancient Mathematicians: Many mathematicians have given Rsin θ values for different angles and also methods for getting these values. In Karanapaddhathi (6-8), Putumana Somayaji gives the following values in relation with the rasies (signs) in zodiac, thus:

त्रिज्यार्धमेकराशिज्या त्रिज्यावर्गार्धतः पदम् ।
भवेदध्यर्धराशिज्या ताभ्यामन्यगुणान् नयेत् ॥

Thrijyaardhamekaraasijyaa thrijyaavargaardhatha: padam
bhavedadhyardha raasijyaa thaabhyamanyagunaan nayeth

When half of the trijya (radius) is taken, it is the chord of the arc of 30° (Rsin 30°). The square root of half of the square of trijya is the chord of 1 rasi (i.e. R sin 45°) From these two the chord of other angles can be calculated (one rasi = 30):

i.e. Radius = Trijya = 3438 R sin 30° = 3438/2 = 1719′

$$R\sin 45° = \sqrt{(3438)^2/2} = 3438/\sqrt{2} = 2431'$$

Prof. T. A. Sarawathy [72] has derived this equation from Bhaskaracharya II

$$R\sin 18° = \sqrt{5R^2} - R/4 \quad R\sin 36° = \sqrt{5R^2 - \sqrt{5}\,R^4/8}$$

Lallacharya, a renowned astronomer of ancient India, has given the following values in Sishyadhi vruddhi Tantra. (2-8,9).

...... वस्वनलाब्धिवन्हयः इमां त्रिभज्यामथ चक्रलिप्तिका ....
जिनाश नगगोगुणेन्दवो ....

... vasvanalaabdhivanhaya: imaam thribhagyaamatha
chakralipthikaa....... jinaamsa nagagogunendavo..........

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It is said that 3438 is Rsine (Jya) of an arc of 90°, this is also radius of circle (in radians) where circumference is 360° or 21600′.... and R sine of an arc of 24° is 1397′.

Relations among R sin θ, R cos θ and R tan θ: The arc and its radius, angle, etc and the sine and cosine are given by mathematicians and astronomers. Quoted here is that explained by Neelakanta Somayaji in his Aryabhateeya bhashya (I. 48-50)

एकचापसमस्तज्या श्रुतिरूपाखिलेष्वपि चापभागेष्विहेच्छास्यान्मानं
व्यासदले तथा तत्तत्कार्मुकमध्याग्रे कोटिदोर्ज्ये फले उभे इच्छाफले
तु दोः कोट्योः खण्डज्ये ज्ञेयता ययोः त्रैराशिकं द्वयं कार्यं चापे
चापे तयोश्च तैः ।।

Ekachaapa samasthajyaa sruthiroopaakhileshvapi
chaapabhageshvihecchaa syaanmaanam vyaasadalam thathaa
thatthath kaarmuka madhyaagre kotidorjye phale ubhe
icchaaphale thu do. kotyo: khandajye jneyathaa yayo.
thraıraasikam dvayam kaaryam chaape chaape thayoscha thai:

The chord of the elemental arc in the form of a hypotenuse is needed throughout. The radius is the argument in the same way. The cosine and the sine at the mid point of the elemental arc are the two fruits. The two processes are the sine and cosine differences. These two rules of three should be applied arc to arc.

Karanapaddhathi (8-8) also gives an important relation between Rsin θ and Rcos θ.

अन्योन्यकोटिहतयोरभिमत गुणयोस्त्रिजीवया हृतयोः ।
योगवियोगौ स्यातामभिमतगुणचापयोगविवरगुणौ ।।

Anyonya kotihathayorabhimatha gunayosthrijeevayaa hathayo:
yogaviyogow syaathaamabhimathagunachaapa yogavivaragunow

The sum of the products of Sin A and Cos B and when

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angles are exchanged, Sin B and Cos A, gives the Sin of the sum of the angles. Similarly the difference of the above gives the value of the sin of angular difference. I.e

$$\sin(A+B) = \sin A \cos B + \cos A \sin B \qquad \text{And}$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B.$$

Formulation of this equation needs great mathematical talent to calculate the sine, cosine and the complex rules related with their addition and multiplications. Earlier to Puthumana Somayaji, Sangamagrama Madhavacharya has also given this equation in Yuktibhasha. Madhava's explanation has the same meaning, given in different wordings. In Varahamihira's Panchasiddhantika (4, V. 19), another set of complex relations among the sine and cosine angles are given which can be mathematically summarised as :

$$(R \sin\theta)^2 = R/2\,(R - R\cos 2\theta)^2.$$

$$R^2 \sin^2\theta^2 + R^2\cos^2\theta^2 = R^2$$

It is important to note that Varahamihira's period is nearly one and half a millennia ago. In Vateswara siddhanta (II.i.57) the followiong equations are given, in 904 AD.

क्रमगुणकृतिर्विभक्ता स्वोत्क्रममौर्व्या फल त्रिभज्योनम् वान्यः
कोटिभुजांशैस्त्रिभाद् विहीनाद्गुणोवान्यः ।।

Kramaguna kruthirvibhakthaa svolkrama mourvyaa
phalam thribhajyonam vaaanya: kotibhujaamsai
sthribhaad viheenaadgunovaanya:

The square of the Rsine of bhuja or koti divided by its own R versed sine and the result diminished by Radius gives the other (R sine or Rcos). Rsine of 3 sign (90°) minus the degree of koti or bhuja is also the other.

These equations can be mathematically presented as follows:

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$R \cos\theta = (R \sin\theta)^2 / R \text{ verse } \theta - R$

$R \sin\theta = (R \cos\theta)^2 / R \text{ verse } (90 - \theta) - R$

$R \sin\theta = R\sin(90\text{-koti})$ or $R \sin(90^\circ - (90^\circ - \theta))$

$R \cos\theta = R \sin(90^\circ - \text{bhuja})$ or $R \sin(90^\circ - \theta$ (R is radius and θ is angle)

Vateswara siddhanta II.i.56 (page 116) also gives the following relations of the radius, sinθ and cosθ

त्रिज्याबाहृवग्रमौर्व्यो: कृतिविवरपदे वेतरज्या प्रदिष्टा
बाहृवग्रज्या (त्रि) मौर्व्योर्विवर युतिहतेर्मूलमाहुस्तयोर्वा
व्यस्तज्या-व्यस्तजीवा विरहित निहतेर्यत्पदं स्यात्क्रमज्या
व्यासघ्ना व्यस्तजीवा निजकृतिरहिता मूलमस्या: क्रमज्या

Thrijyaabaahvagra mourvyaa: kruthivivarapade vetharajyaa pradishtaa baahvagrajyaa (thri) mourvyorvivara yuthi hathermoolamaahu sthayorvaa vyasthajyaa-vyasthajeevaa virahitha nihather yathpadam syaathkramajyaa vyaasaghnaa vyasthajeevaa nijakruthirahithaa moolamasyaa: kramajyaa

This can be mathematically represented as follows:

(Kotijya) $R \cos\theta = \sqrt{R^2 - (R\sin\theta)^2}$

(Bhujaya) $R \sin\theta = \sqrt{R^2 - (R \cos\theta)^2}$

$R \cos\theta = \sqrt{R (R - R \sin\theta) (R + R \sin\theta)}$

$R \sin\theta = \sqrt{(R - R \cos\theta (R + R \cos\theta)}$

Lallacharya in Sishyaddhi vruddhi Tantra (3-3) has given another rule for sine and cosine equations.

दोर्ज्यावर्ग विवर्जित त्रिभवनज्या वर्गमूलं भवेत्
कोटिज्या भुजभागवर्जित नवत्यंशोत्थजीवाथवा
स्पष्टस्वस्वगुणाहते खव्रसुभिर्दो: कोटिजीवे हृते

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स्यातांदोःफलकोटि संज्ञाकफले ताभ्यां श्रुतिं साधयेत् ॥

Dorjyaavarga vivarjitha thri bhavanajyaa vargamoolam bhaveth
kotijyaa bhuja bhaaga varjitha navathryamsotthajeevaathavaa
spashta sva svagunaabathe khabrasubhirdo: kotigeevebruthe
syaathaam do: phalakoti samjnakaphale thaabhyaam sruthim saadhayeth

Square root of the difference of squares of R sine 90° and R sine of an arc, is R cosine of arc OR R sine of the difference of 90° and the arc is R cosine of arc.

In Karanapaddhathi (6-10) an equation relating R sin θ, R cos θ and arrow is given.

यद्वेष्टचापगुणतच्छरवर्गयोगमूलार्धमिष्टधनुरर्धगुणः प्रदिष्टः ।
ज्यानांनिजत्रिगुणवर्गविशेषमूलंकोटिस्तदूनसहितौत्रिगुणौस्वभाणौ ॥

Yadveshta chaapagunatha ccharavargayoga
moolaardhamishtadhanurardhaguna: pradishta:
jyaanaam nijathriguna vargaviseshamoolam
kotisthadoona sahithow thrigunow svabhaanow

Square root, of the square of a chord (R sin θ) diminished from squares of radius gives the koti (R cos θ). This subtracted from radius gives the (small) arrow of arc. This added to radius is big arrow of the arc.....

In Karanapaddhathi another relation is given (6-11)

यद्वेष्टकोट्याहतविस्तरार्धेनोनान्वितौ व्यासदलस्य वर्गौ ।
अर्धीकृतौ तौ पदितावभीष्टचापार्धदोः कोटिगुणौ भवेतां ॥

Yadveshtakotyaahathavistha raardhenonaanvithow
vyaasadalasya vargow ardheekruthow thow padithaa
vabheeshtachaapaardhado: kotigunow bhavethaam

The R cos θ when multiplied with radius and diminished the result, from the square of the radius, Square root of half of

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this will give the R sin θ/2 (instead diminishing) if added, the result to the square of radius and followed the procedure then koti will be the result i.e R cos θ. Mathematically it is written as:

$$\sqrt{\tfrac{1}{2}(R^2 - R^2\cos\theta)} = \quad R\sin\theta/2 \text{ similarly}$$

$$\sqrt{\tfrac{1}{2}(R^2 + R^2\cos\theta)} = \quad R\cos\theta/2$$

The theorem given by Puthumana Somayaji on the arc, chord and sin and cos angles (Karanapaddhathi 6-12, 13a) is of great significance.

चापाच्च तत्तत् फलतो/पि तद्वच्चापाहताद्द्व्यादि हतत्रि मौर्व्या
लब्धानि युग्मानि फलान्यधो/धश्चापाद्युग्मानि च विस्तरार्धात्
विन्यस्यचोपर्युपरित्यजेत् तच्छेषौ भुजाकोटिगुणौ भवेताम् ॥

Chaapaaccha thatthath phalathoapi thadvath
chaapaahathaadvyaadi hathathri mourvyaa labdhaani yugmaani
phalaanyadhodhaschaapaadayugmaani cha vistharaardhath
vinyasyachoparyuparithyajeththaccheshow bhujaakotigunow bhavethaam

The arc is multiplied by itself (for a definite number of times) and the product is multiplied once again by the same arc. The result is divided by the product obtained by multiplying 2, 3, ... etc., (upto the same definite number) with the radius repeatedly multiplied by itself (upto the same number of times). The quotients thus obtained for the even values of definite numbers (upto which the aforesaid multiplication is repeatedly done) are set one below the other (in one column) and likewise those corresponding to the odd values of (definite) numbers (in another column). From the first terms is subtracted the term immediately below and so on for every column. All the values thus obtained in the case of the even column are subtracted together from the arc. Similarly the corresponding values for the odd column together subtracted from the radius. The results are the bhujajya (R sine θ) and kotijya (R θ) respectively.

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This equation can be summarised as follows after carrying out the procedure directed in the theorem.

$An - S^n \times S/(2 \times 3 \times 4 \times \ldots (n+1)) \times r^n : (S = arc) = S^{(n+1)}/(n+1)!r^n$
The results obtained are bifurcated as odd numbers and even numbers. Odd number results are labelled as A1, A3, A5...... And even number A2, A4, A6....

When n is substituted by the respective odd or even numbers the following separate schedule will be obtained for odd series and even series.

(A1-A3), (A5-A7), (A9-A11), ...

(A2-A4), (A6-A8), (A10-A12), ... From the above,

Bhujajya or $R \sin\theta = S$ - (A2-A4) - (A6-A8) - (A10-A12),.....

Kotijya or $R \cos\theta = R$ - (A1-A3) - (A5-A7) - (A9-A11),......

From these equations the modern forms can be obtained by substituting value of $S = R\theta$ for small values of S and as follows:

$\sin\theta = \theta - \theta^3/3! + \theta^5/5! - \theta^7/7! + \ldots$ and

$\cos\theta = \theta - \theta^2/2! + \theta^4/4! - \theta^6/6! + \ldots$

Where 3! is (pronounced as factorial 3) 1 x 2 x 3 and same for others too. In the above, Sn is the arc, θ is the angle and R is the radius.

In Karanapaddhathi an interesting example is given with the help of an ancient commentary:

Multiply 5400′, (angle of 3 rasi - 90° given in minutes) with itself and divide by twice the radius (in minutes) 3438′. The result is 4241′9″-0‴. This is odd value (in minutes, seconds and one tenth of the seconds).

Multiply the above result with 5400′ and divide by thrice

the radius. The value available is 2220′ 39″ 40‴. (this is for even value)

This value is multiplied with 5400 and divided by thrice the radius. Corresponding odd value obtained is 872′ 3″ 5‴.

When the procedure is repeated, the next even value obtained will be 273′ 57″ 47‴.

And the next odd value obtained will be 71′ 43″ 24‴.

The next even value will be 16′ 5″ 41‴, the next odd values 3′ 9″ 37‴. Next even value is 0′ 33″ 6‴, and odd value 0′ 5″ 12‴. Next even value is 0′ 00″ 44‴ and last odd value is 0′ 0″ 6‴

By arranging the values according to the directions given in the theorem, sine and cosine value will be obtained. All the angles given above are mentioned in katapayadi number system and in the above explanations only their equivalents are given. The series was developed from the preliminary theorem put forth by Madhavacharya (1400 AD) on the sine, series. This theorem in modern mathematics was rediscovered by James Gregory (1638-1675 AD). The mathematicians recognised the original contributions of Madhavacharya and hence this series is now renamed as Madhava-Gregory series. What is followed in the above series by Puthumana Somayaji is another novel series discovered by him. Somayaji was also anterior to James Gregory. Henace this "Gregory series" is really "Puthumana Somayaji Series". However the father of sine, cosine series of equations and theorems is Madhavacharya.

Tangent of angles: Sin θ divided by Cos θ gives the Tangent θ. It is based on the sides of a Rt angled triangle. These parameters are widely used in modern mathematics for calculating slopes. Slope of any structure is directly related to its tangent. Available literaure does not give any details on ancient Indian contribution

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on this geometrical parameter of tan θ. Hence presentation of this data calls special significance because, it is the revelation of a new topic on which Indians had made their original contributions. The theorem given in Karanapaddhathi (6-18) is of significance in this context.

व्यासार्धेन हतादभीष्टगुणतः कोट्याप्तमाद्यं फलं
ज्यावर्गेण विनिघ्नमादिमफलम् तत्तत्फलं चाहरेत् ।
कृत्वा कोटिगुणस्य तत्रतु फलेष्वेकत्रिपञ्चादिभि-
र्भक्तेष्वोजयुतैस्त्यजेत् समयुतिं जीवाधनुःशिष्यते ।।

Vyaasaardhena bathaadabheeshtagunatha kotyaaptha maadyam phalam
jyaavargena vinighna maadimaphalam thatthathphalam chaahareth
kruthvaakotigunasya thathrathu phaleshvoekathri panchaadibhir
bhaktheshvoojayuthais thyajeth samayuthim jeevaadhanu: sishyathe

Jya of an arc to be multiplied by radius and divided by koti (R cos θ). This is the first term (of the series) The value of the first term when multiplied by the square of Jya (R sin θ) and divided by square of the koti gives the second term. This process is repeated. The successive terms are divided by the odd integers 1, 3, 5, ...Now, when the consecutive terms in the series starting from the first term are alternatively subtracted and added gives the circumference of the arc.

This theorem can be summarised in mathematical terms as:

First term = $R \times R \sin\theta / R \cos\theta = R \tan\theta$

Second term = $R \times R \sin\theta \, R \cos\theta \times (R \sin\theta)^2 / (R \cos\theta)^2$

$= R \tan\theta \times (\tan\theta)^2 = R (\tan\theta)^3$

The third term will be obtained by multiplying the second result with the second part of the second equation, and it will be equal to R $(\tan\theta)^5$ and so on for further results. Alternatively adding and subtracting, after dividing with 3, 5, 7,..... the following

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equation will be obtained. When $S = R\theta$ or $\theta = S/R$ i.e.

$\theta = \tan\theta - 1/3\,(\tan\theta)^3 + 1/5\,(\tan\theta)^5 - \ldots\ldots$ This formula holds good when the angle is equal to or greater than 45. Where S is the arc and θ the angle and R, the radius of the arc. This theorem even though discovered by Puthumana Somayaji in 1438 AD is at present known as De Moire's (1667 AD) theorem.

$S = r\,(r\sin\theta/r\cos\theta) - 1/3\,(r\sin\theta^3/r\cos\theta^3)\ 1/5 + (r\sin\theta^5/r\cos\theta^5) - \ldots\ldots)$ When $r\sin\theta/r\cos\theta$ is taken it is $\tan\theta$.

Sankara Varman[72] gave another theorem which is mathematically written as below:

व्यासघ्नेऽर्कपदे कृतेऽग्निभिरेतानिते च तत्फलात्
चैकैकद्ययुजाहृतेषु परिधेर्भेदो युगोनैक्ययोः
एवं चात्र परार्धविस्तृतिमहावृत्तस्य नाहोक्षरैः
स्याद् भद्राम्बुधिसिद्धजन्मगणितश्रद्धास्मजभूपगीः ।।

Vyaaaghnearkapade krutheagni bhiirethaanithe chathathphalaath
chaikaikadyayujaahrutheshu paridherbhedo yugonaikyayo:
evam chaathra paraardha vishthruthimahaa
vrutthasya naahoksharai. syaad bhadraambudhi siddhaj
anmaganitha sraddhasma ja bhoopagee:

This can be mathematically represented as

$\Pi = 6\tan(1/\sqrt{3}) = 6\,(1/\sqrt{3} - (1/\sqrt{3})^3/3 + (1/3)^5/5 - \ldots\ldots$ ($\tan^{-1}$ is tan inverse) This is also known as Gregories series, after the name of the 'discoverer' Gregory (1670 AD). But Sankaravarman discovered this theorem in 1535 AD. Sankaravarman also gave an extension to this equation as mathematical presentation is as given in the reference.

$\Pi = \sqrt{12}/1 - \sqrt{12}/(3\times3) + \sqrt{12}/(3^2\times5) - \sqrt{12}/(3^3\times7) + \sqrt{12}/(3^4\times9) - \ldots$

Volume of geometrical figures: The volume of any geometrical

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figure is the content of material present in that structure. Amount of water contained in vessel when it is filled, is the inner volume of the vessel. Similarly for a cube, globe, pyramid, cone, etc, and all type of structures volume can be determined. Archimedes found out the volume by dipping the material in a bigger vessel containing water upto the brim. The quantity of water overflowed is the volume of the structure. Volume is calculated using mathematical equation if it has a uniform shape. A cube, a cone, sphere, hemisphere, etc. are uniform structures. The volume of these, can be calculated directly. In ancient India the procedures adopted for calculating the volume of some structures were accurate. Whenever, the level of accuracy was not attained, the next generations of mathematicians have tried to attain perfection in the values and methods. Method for the determination of volumes are given in Ratna pareeksha and Rasa sastra. Finding out the volume was adopted as a method for examining purity of gold and jewels. Some other books also casually describe the volume of structures. A few descriptions can be seen in Puranas and Artha sasthras. Narada purana (purva bhaga, II, 54, 51-56) gives the explanation for determining the volume of water reservoirs.: The product of length, breadth and depth (in angulas) of a reservior would be the quantity of water in the reservoir, in Dronas (unit of measurements). Similarly, to find out the weight of food grains (in drona) in a heap, its length, breadth and height (in angulas) are multiplied and divided by 96. The length, breadth and thickness of a piece of stone (in angulas) multiplied and then divided by 1150 would give its weight in Dronas. Similarly, the length, breadth and thickness of a piece of iron (in angulas) multiplied, and divided by 585 would give its weight in Dronas. (The rule has to be examined for ascertaining the accuracy). However the concept of volume, the density of the matter and finding out the weight, were known to the authors of the Narada

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purana and to the people of that time [11], which is supposed to be written not later than 200 BC. This tradition of finding out the quantity of materials having different shapes was continued and correct equations were derived in due course of time.

Volume of uniform structures : Brahmasputa siddhanta gives the general equation for determining the volume of structures when they have a uniform pattern In Ganithadhyaya of this book (XII. 45, 46), it has been stated as follows:

मुखतलयुतिदलगुणितं वेधगुणं व्यावहारिकं गणितम्
मुखतलगणितैक्यार्धम् वेधगुणं स्याद् गणितमौत्रम् ।
औत्रगणिताद्विशोध्य व्यावहारफलं भजेद् त्रिभि:
शेषम् लब्धम् व्यावहारफले प्रक्षिप्य भवति फलं सूक्ष्मम् ।।

Mukhathalayuthi dalagunitham vedhagunam vyavahaarikam ganitham mukhathalaganithaikyaardham vedhagunam syaad ganithamouthram authraganithaadvisodhya vyavahaara phalam bhajed thribhi: sesham labdham vyavahaaraphale prakshipya bhavathi phalam sookshmam

Practical volume to be equal to the area found by taking mean of the linear dimension of top and bottom and multiplied by the height. Gross volume to be equal to the mean of the area of the top and bottom multiplied by the height.

Even though Brahmagupta has given two methods for determining volume, for a uniform structure, the product of its surface area and height will give the volume. Part of this rule is applicable when the lower and upper surface have different measurements but almost same shape. When there is a variation in the surface area, the measurement has to be taken as follows: If the material has the shape of a cylinder, then, Practical volume = $\Pi (r_i + r_i)/2)^2 \times h$ and Gross volume = $(\Pi r_i^2 + \Pi r_{ii}^2)/2 \times h$ The upper radius is r_i and lower radius is r_{ii} and the height is the same for both.

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In Lilavati (page 304-3a) the universal rule for determining the volume of a straight non pointed structure is given·

क्षेत्रफलं सममेवं वेधहृतं घनफलं स्पष्टम् ।

Kshethraphalam samamevam vedhahrutham ghanaphalam spashtam

Area multiplied with average depth gives volume of the figure. This gives an approximate answer equal to the modern value, because average height gives near approximation.

In Lilavati (p 310-1) application of this rule is correctly followed as the product of surface area and height is the volume of the structure, in this example

उच्छ्रयेण गुणितं चितेः किलक्षेत्रसम्भवफलं घनं भवेत्
इष्टिकाघनहृते घने चितेरिष्टिकापरिमितिश्च लभ्यते
इष्टिकोच्छ्रय हृदुच्छितिश्चितेः स्युः स्तराश्च दृषदो चितेरपि ।।

Ucchrayena gunitham chithe: kilakshethra sambhava phalam ghanam bhaveth ishtikaaghanahruthe ghane chitherishtakaa parimithascha labhyathe ishta kecchraya hruducchithischithe: syu: stharaascha drushadaam chitherapi

(When bricks are arranged in a platform - chiti - in length breadth and height-) the area of the upper surface of chiti multiplied with height gives volume of the chiti. When the volume of chiti is divided by the volume of one brick, the number of bricks in it can be obtained. Similarly, length of chiti divided by length of bricks gives, the number of bricks in length and that is applicable for (the number of bricks in) height layer and base layer.

In Lilavati an exercise is given (page 308 rule 3 and example 2)

खातेऽथ तिग्मकरतुल्य चतुर्भुजे च किंस्यात् फलं नवमितः किल यत्र वेधः। वृत्ते तथैव दशविस्तृतिपञ्चवेधे सूचीफलं वद तयोश्च पृथक्पृथक् मे ।।

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Khathe fatha thigmakarathulya chathurbhuje cha
kimsyaath phalam navamitha: kilayathra vedha:
vruttbe thathaiva dasavisthruthi panchavedhe
soocheephalam vada thayoscha pruthak pruthak me

Find out the volume of a rectangular structure with square face of 12 unit length and 9 unit depth. Similarly find out the volume of a cylinder with diameter 10 unit and depth 5. Also in the above two forms, find out the volume of pyramid (in former) and cone (in latter) with above dimensions respectively.

These four problems are fine examples to show that relations among the volumes of a rectangular structure and the pyramid, and the cylindrical structure and the cone were well known. Another problem on the volume of a triangular pyramid is given in the Bhaskara bhashyam for the Aryabhateeya (58.1)

श्रृङ्गाटकघनगणितम् द्वादशगणिताश्रितस्य यच्चास्य
ऊर्ध्वभुजापरिमाणं स्फुटतरमाचक्ष्व मे शीघ्रम् ।

Srungaatakaghanganitham dwaadasaganithaasrithasya
yachhaasya oordhwabhujaa parimaanam sphutathara
maachakshwa me seeghram.

Quickly tell me the more accurate volume and also the measure of the altitude of the solid having the shape of a trapa in which each edge is 12 units

The method adopted by Bhaskara is not correct since he used the same wrong formula used by Aryabhatta.

Volume of conical structure: The volume of a conical structure is given accurately in Lilavati (304 P. 3b) by Bhaskaracharya II

समखातफलत्र्यंशो: सूचीखते फलं भवति ।

Samakhaatha phalathryamasai: soochikhathe phalam bhavathi

The one third of the volume of the uniform cylinder is the volume of the cone.

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The volume of the cylinder is known as the product of the area of the circular surface and the height, this formula correctly defines the volume of the cone. Earlier to Bhaskarachrya II, Brahmagupta in Brahmasphuta siddhanta (XII.44) has explained the volume of the cone, while defining the volume of a conical well, for estimating the quantity of the soil to be removed from it during digging. The translation of the statement is: Volume of a uniform excavation divided by three is the volume of the needle shaped solid'. Needle shaped solid is the cone, one third of the volume of cyliner, is the volume of the cone, having the same height and radius. This shows that the equation for finding out volume of a cone is derived at least during the 6th century AD.

Volume of partial cones: A conical structure, can be partial also having one half, one fourth or three fourth of the full cones. These structures or shapes are common too. An interesting applied problem on cones is given by Bhaskaracharya II, for calculating the volume of grains kept in different types of partial and full cone shapes (Lilavati 317 Page)

परिधिर्भित्तिलग्नस्य राशेस्त्रिंशत्करः किल अन्तकोणस्थितस्यापि
तिथितुल्यकरः सखे । बहिष्कोण स्थितस्यापि पञ्चघ्ननव सम्मितः
तेषामाचक्ष्व मे क्षिप्रम् घनहस्तात् पृथक् पृथक् ॥

Pandhirbhitthilagrasya raasesthrimsathkara: kila anthakonasthithasyaapi thithithulyakara: sakhe bahishkona sthithasyaapi panchaghnanava sammitha: theshaa ma dhakshva me kshipram ghanahasthaath prathak prathak

Friend, the food grains are kept at a circumference of 30 cubit in the floor, outside corner of the room, inside corner and side of the wall. Find out the volume of the grain if the height is 45 cubit.

When food grains are kept on the floor without the support of wall, equation for the volume of the cone can give, the volume

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of food grain. In the inside corner of outside wall onefourth of a cone gives the volume of grain three fourth of the volume of cone will be the volume of grain, on outside of outside wall, and half the original volume will be the volume of grain supporting on a wall.

The rules applied by Bhaskara II, given below, is only an approximation not the exact formula (Lilavati 414 Page, 1b)

भवति परिधिषष्ठे वर्गिते वेधनिघ्ने घनगणितकराः
स्युर्मागधास्ताश्च स्वार्यः ।

Bhavathi paridhishashte vargithe vedhanighne
ghanaganitha karaa: syurmaagadhaasthaascha svaarya:

The square of one sixth of the circumference multiplied with the height gives the volume of food grain; This method was followed by the people of Magadha.

According to this, the formula is $\Pi \times \Pi \times r \times r \times h/9$. Where as the correct formula according to modern mathematics is $\Pi \times r \times r \times h/3$ (i.e 0.9554 x Bhaskara's value). However the definition given by Bhaskara is perfectly correct(Lilavati 304p.3b)

Volume of spheres: Aryabhatta I had given a method for the determiantion of the volume of spheres (Aryabhateeya 2.7)

समपरिणाहस्यार्धम् विष्कम्भार्धहतमेव वृत्तफलं ।
तन्निजमूलेन हतं घनगोलफलं निरवशेषम् ।।

Samaparinaahasyaardham vishkambhaardha hathmeva vruthaphalam
thannijamoolena hatham ghanagolaphalam niravasesham

The area of a circle is the product of half the circumference and half of the diameter. This multiplied by its own square root gives the exact volume of the sphere i.e $\Pi r^2 \times \sqrt{\Pi r^2}$, will be the volume of a sphere of radius r.

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This equation will yield only approximate answer. Mahaveeracharya (805 AD) has given another formula in Ganita sara sangraha (VIII. 28a) in which the volume of sphere is stated as 9/4 x 9/4 x r^3. Bhaskara I gives the volume as 9/2 x r^3. It appears that even upto Prutudaka (860 AD), who wrote a commentary on Brahmasphuta siddhanta (XI. 20), many mathematicians used the same equation given by Aryabhatta I for calculating the volume of a sphere, whereas Sreedharacharya (900 AD) used the equation 4 (1+1/18) r^3 which is given in Trisatika (rule 56). Aryabhatta II (950 AD) in Mahasiddhanta (XV 108) and Sreepati (1039 AD) in Siddhanta sekhara (XIII 46) used this equation for calculating the volume of spherical bodies. Bhaskara II gave the correct formula for the volume of a sphere, (Lilavati 201 (c-d)).

वृत्तक्षेत्रे परिधिगुणितव्यासपादः फलं तत्क्षुण्णं वेदैरुपरिपरितः
कन्दुकस्येवजालम् । गोलस्यैवं तदपि च फलं पृष्ठजं व्यासनिघ्नम्
षड्भिर्भक्तम् भवति नियतम् गोलगर्भे घनाख्यं ॥

Vrutthakshethreparidhigunitha vyaasapaada: phalam
thathkshunnam vedairupari paritha: kandukasyevajaalam
golasyaivam thadapi cha phalam prushtajam vyaasanighnam
shadbhirbhaktham bhavathi niyatham golagarbhe ghanaakhyam

When the circumference is multiplied with the diameter and that result divided by 4, gives the area of a circle. This when multiplied with 4 gives the surface area of the globe like that of a ball. Further multiplied with diameter and divided by 6, one gets the volume of that sphere. $4 \Pi r^2 \times 2 r \times 1/6 = 4/3 \Pi r^3$

This is the same as the modern equation. In short, values given by Mahavira is 4.22 x r^3, Aryabhatta I is 5.56 x r^3 Bhaskara I is 4.5 x r^3 and the correct value is 4.187 x r^3.

Mathematical theorems discovered by Indians:

A number of mathematical theorems have been discovered

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by ancient Indian mathematicians some of which have already been explained. Many of these Indian theorems are at present known in foreign scholars names. Some of them also remain unknown. No effort appears has been taken to prove that these are originally Indian contributions to modern mathematics. With an exception of the work done by Dr. K.V. Sarma[73] which stands unique. Literally, he has brought the facts and figures to put forth the authentic claims on Indian achivements, in the development of these theorems. Some of these Indian theorems are given below, with its foreign names.

Gupta [74] in his book on 'Second order interpolation in Indian mathematics upto fifteenth century says that Newton Stirling interpolation formula was known to Brahmagupta. Following the rules of Brahmagupta, Kerala astronomer Govindaswamin (800 AD) had given a set of rules for the comutation of intermediary functional values. Govindaswamin gives this in his commentary to Mahabhaskareeya (4.22).

Newton Gauss Interpolation formula of Govindaswamin:

गच्छाद्यात गुणान्तरवपुर्यातैष्य दिश्वासनाच्छेदाभ्यास
समूहकार्मुककृतिप्रापतात् त्रिभिस्ताडितात्
वेदैर्हिषड्भिर् अवाप्तम् अन्त्यगुणजे राश्योः क्रमात् अन्त्यभैः
गन्तव्याहत वर्तमानगुणजाच्छापाप्तम् एकादिभिः ।
अन्त्याद् उत्क्रमतः क्रमेण विषमैः सङ्ख्याविशेषैः
क्षिपेद्भङ्क्त्वाप्तं, यदि मौर्विकाविधिर् अयम् मख्याः
क्रमात् वर्तते शोध्यम् व्युत्क्रमतास्तथाकृतफलम्.....

gacchad-yata-gunantharavapuryathaishya-disvasanaac-
chedaabhyaasa-samuha-kaarmukakrti-praapthath tribhisthaadithah
vedaihi sadbhir avaaptam antyagunaje rasyo: kramad antyabhe
ganthavaahata-varthamaana-gunajaacchapaatham ekaadibhi:

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antyad utkramatah kramena vishamat: sankhyavisesbat: khsiped bhanktiruaptam, yadi mastvikasadher ayam makhyah kramad vartate sodhyam vyutkramathaastathakrthaphlam.....

Mathematicaly this formula is summarised as follows:
$F(x+nh) = \Delta f(x) + nf(x) + \frac{1}{2}n(n-1)\ (\Delta f(x) - \Delta f(x-h)$

Multiply the difference of the last and the current sine differences by the square of the elemental arc and further mutiply by three. Now divide the result so obtained by four in the first rasi, or by six in the second rasi. The final result thus obtained should be added to the portion of the current sine difference (got by linear proportion). In the last rasi, multiply the linearly promotional part of the current sine differences by the remaining part of the elemental arc and divide by the elemental arc. Now, divide the result by the odd numbers according to the current sine difference, when counted from the end in the reverse order. Add the final result thus obtained to the portion of the current sine difference. These are the rules for computing true sine differences for sines. In the case of versed sines, apply the rules u the reverse order and the above corrections are to be subtracted from the respective differences.

This interpolation formula is now known as the Newton Gauss (1670 AD) interpolation formula as mentioned by Whittaker and Robinson, It is actually the contribution of Govindaswamin.

Newton Gauss backward interpolation formula of Vatesvara: This interpolation was known to Brahmagupta, which is given in Khandakhadhyaka (II.i.4). Brahmagupta's period was 1000 year before Newton. It is also given by Vateswara in Vateswara siddhanta (II.1.66). Quoted here is from Vateswara:

धनुषाप्त भुक्त जीवाघाते लब्धम् सरूपकं दलितम् ।
लब्धघ्न विवरहतम् च संशोध्य नियोज्य विकलज्या ।।

Dhanushaaptha bhuktha jeevaghaathe labdham saroopakam dalitham
labdaghna vivarahatham cha samsodhya niyogya vikalayaa

In modern mathematical form this interpolation formula can be written as $f(x) = f(x_i) + (x-x_i)1/h\,\Delta f(x_i-h) + (x-x_i)1/h.\,(x-x_i+h)1/h.\,\Delta^2 f(x_i-h)\tfrac{1}{2}$.

Add 1 to the labdha (what is obtained on dividing the residual arc by the elemental arc), reduce it to half, and then multiply that by the product of the labdha and the vivara (that is the difference between the traversed and untraversed Rsine differences). Subtract that from or add that to the product of the labdha and the traversed Rsine difference. Then is obtained the residual Rsine difference.

Many modern mathematicians are aware that Newton Gauss forward interpolation formula was known to Indians, but they never believed that the backward interpolation formula was known to Indians. Vateswara siddhanta gives clear proof for the knowledge of that formula also. Brahmagupta was the first mathematician to give this backward interpolation formula. But Brahmagupta's formula is now known as Striling's formula of interpolation.

Taylor series of sine and cosine function of Madhavacharya: This function has been described to Sankara Variyar's explanation on Nilakanta's Tantrasangraha (II, 10-13). The stanzas given below are from Nilakanta Somayaji's commentary of Aryabhateeya (2-12)

इष्टदो: कोटि धनुषो: स्वसमोपसमीरते ज्ये द्वे सावयवे न्यस्य
कुर्याद् ऊनाधिकम् धनु: द्विघ्न तल्लिप्तिकाप्तैक शरशैल शिखीनन्दघ:
न्यस्याच्छेदाय चमिथास्तत्संस्कार विधित्सया चितवैकाम् प्रक्षिपेज्जह्यात्
तद्धनुष्यधिकोणके अन्यस्याम् अथ ताम् द्विघ्नां तथाऽस्याम्
इति सस्कृति: संत ते कृत संस्कारे स्वगुणौ धनुषास्तयो:

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ista-dohkotidhanushoh swasamipasamirate jye dve saavayave nyasya kuryaad unaadhikam dhanuh dvighna tallipokaptakasansailaakhndaiah nyasyacchedaarya cha mithastatsamskaaravidhitsaya anyasyam atha tuam dvighnaam tathaa syam iti samskriti. santha te krtasamskare swagunau dhanusas tayo:

Placing the sine and cosine chords nearest to the arc, whose sine and cosine chords are required, get the arc difference to be subtracted or added. For making the correction, 13,751 should be divided by twice the arc difference in minutes and the quotient is to be placed as the divisor, divide the one (sine or cosine) by this divisor and add to or subtract from the other (cosine or sine) according as the arc difference is to be added or subtracted. Double this result and do as before. Add or subtract the result to or from the first sine or cosine to get the desired sine or cosine chords.

It is found that this theorem is from Madhava of Sangamagrama (1340 AD). Somayaji has mentioned that this is from Madhava's book. But it is now known as the Taylor series of sine and cosine functions discovered by Book Taylor ((1685 AD)

Newton power series of Madhavacharya: Newton series of this nature was given by Madhavacharya in Yuktibhasha and also by Puthumana Somayaji in Karanapaddhathi (6-12,13)

निहत्य चापवर्गेण चापम् तत्तत् फलानि च हरेत्
समूलयुग्वर्गैस्त्रिज्या वर्गहतैः क्रमात्
चापे फलानि चाधोधोन्यस्योपर्युपरि त्यजेत् जीवाप्त्यै,
संग्रहो/ स्यैव विद्वान् इत्यादिनाकृतः
निहत्य चापवर्गेण रूपं तत्तत् फलानि च हरेद्
विमूलयुग्वर्गैस्त्रिज्यावर्ग हतैः क्रमात्
किन्तु व्यासदलेनैव द्विघ्नेनाद्यम् विभज्यताम् फलान्यधोधः
क्रमशोन्यस्योपर्युपरि त्यजेत्
शराप्त्यै, संग्रहो/स्यैव स्तेनस्त्रीत्यादिनाकृतः

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nihatya chapavargena chapam tatthathphalani cha
haret samulayugvargaistrijyavargahatai: kramaat
chapam phlani chadhodhonyasyoparyupari tyajet
jivaptyai, sangraho syaiva vidvan-ityadina krtha:
nihathya chapavargena rupam tattatphalani cha
hared vimulayugvargaistrijyavargahatai: kramat
kintu vyasadalenaiva dvighnenadyam vibhajyataam
phalanyadhodha: kramaso nyasyoparyupari tyajet
saraptyai, sangrahoa syaiva stenastri-tyadinaa krta:

Multiply repeatedly the arc by its square and divide by the square of even numbers increased by that number and then multiplied by the square of radius. Place the arc and result one below the other and subtract each from what is above it. To derive the arc, which are collected, beginning with the expression *Vidvan* (katapayadi number). Multiply repeatedly, the unit measurement which is the radius, by the square of the arc and divide by the square of even numbers decreased by that number and then multiplied by the square of radius; the first is, however, to be divided by twice the radius. Place the results one below the other and subtract each from the one above it. That is the method to derive the *saras*, which are collected in the beginning with *stena*. This equation is now known as Newton power series.

Lhuiler's formula of parameswara: The formula given in Parameswara's (1360 AD) commentary to Lilavati on the circum radius of the cyclic quadrilateral is known now as the Lhuiler's (1782 AD) formula.

दोष्णाम्द्वयोर्द्वयोर्घातयुतानाम् तिस्राणां वधात्
एकैकोनेतरावैक्यम् चतुष्कवधाभाजितम्
लब्धमूलेन यदवृत्तम् विष्कम्भार्धे निर्मितम् सर्वम्
चतुर्भुजक्षेत्रम् तस्मिन्नेव तिष्ठहते

Doshmamdvayordvayor ghaatayutaanaam tisraanaam vadhaat

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ekaikonetarattraikyam catushkavadhabhajitam
labdhamulena yadvrttam vishkambhaardhena nirmitam
sarvam caturbhujakshetram tasminnevatisthtabathe

The three sums of the product of sides, taken two at a time are to be multiplied together and divided by the product of the sums of the sides taken three at a time and diminished by the fourth. If a circle is drawn with the square root of this quantity as radius, the whole quadrilateral will be situated inside it.

This has been published by Parameswara in his commentary to Lilavati.

Gregory and Leibnitz series for the inverse tangent of Madhava: This series is said to be discovered by British mathematican James Gregory (1638 AD) and in Europe by German mathematician Gottfried Wilhelm Leibnitz (1646 AD). In India this power series was discovered by Sangamagrama Madhava (1350 AD). It is the first of the Tangent series ever known to mathematicians.

It is also given in Kriyakramakari on Lilavati Kanda 2, vrutta 40 and in Yuktibhasha

इष्टज्यात्रिज्ययोर्घातात् कोट्याप्तम् प्रथमम् फलम् ज्यावर्गं गुणकं
कृत्वा कोटिवर्गम् च हारकम् प्रथमादिफलेभ्योऽथ नेया फल कृतिर्मुहुः
एक त्र्याद्योज संख्याभिरभक्तेष्वेतेष्वनुक्रमात् ओजानां संयुतेस्त्यक्त्वा
युग्मयोगम् धनुर्भवेत् दोः कोट्योरल्पमेवेह कल्पनीयम् इहस्मृतम्
लब्धीनाम् अवसानम् स्यान्न तथापि मुहुः कृते

istajya-trijyayorghathath kotyaptam prathamam phalam jyavargam gunakam kritva kotivargam cha haarakam prathamaadiphalebhyoa tha neya phalakrtir muhu. eka-tryaady-ojasankhyabhirabhakteshvetesho anukramaat ojanam samyutesthyaktva yugmayogam dhanur bhavet doh kotyor alpameveha kalpaniyam iha smrtam labdhinam avasanam syanna thathapi muhu: krte

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Obtain the first result of multiplying the jya (R sine θ) by the trijya (radius) and dividing the product by koti (R cos θ). Multiply this result by the square of the jya and divide the square by the koti. Thus we obtain a second result a sequence of the further results by repeatedly multiply by the square of the jya and dividing by the square of the koti Divide the terms of the sequence in order by the odd numbers 1,3,5,.. ; after this, add all the odd terms and subtract from them all the even terms (without disturbing the order of the terms). Thus is obtained the dhanus whose two elements are the given jya and koti. (Here the smaller of the two elements should be taken as the jya, since other wise the series obtained will be non finite)

i.e arc = $R \sin\theta \, Cos\,\theta - R \sin^3\theta/3 \, Cos^3\theta + R \sin^5\theta/5 \, Cos^5\theta....$

Arc = $R \, Tan\,\theta - 1/3 \, R \, Tan^3\theta + 1/5 \, R \, Tan^5\theta - 1/7....$

Or $Tan\,\theta = \theta - a^3/3 + \theta^5/5.... = Sin\,\theta/Cos\,\theta = Tan\,\theta$

Leibnitz power series for Π of Madhava: The power series has been mentioned by the same Kerala astronomer Sangamagrama Madhavacharya. Even though the value of Π has been mentioned in Aryabhateeya and exact value given by Bhaskaracharya II, in Europe the value has been discussed by Lambert in 1671 AD. However the same information has also been given by Nilakanta Somayaji (1444 AD) in his commentary for Aryabhateeya. But Leibnitz (1673 AD) has put forth a series on the Π which is thus mentioned by Madhavacharya about 300 years before Leibnitz:

व्यासे वारिधि निहते रूपहृते व्यास सागराभिहते
त्रिशरादि विषमसंख्याभक्तमृणं स्वम् पृथक् क्रमात्कुर्यात्।।

vyase varidhi-nihate rupahrate vyasasaagaraabhihate tri-sara adi\ vishamasankhya-bhaktamrnam swam pruthak kramat kuryat

Multiply the diameter by 4. Subtract from it and add to it alternately the quotients obtained by dividing four times the

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diameters by the odd integers 3, 5, 7,.... This will give the fine value of the circumference i.e $\Pi/4 = 1-1/3+1/5-....$

The same series is also given by Puthumana Somayaji in Karanapaddhathi, who is also known to be earlier than Leibnitz.

De Moiver's approximation to the value of Π of Madhava: This has been given by Madhavacharya in Yuktibhasha.

यत्संख्याया/त्र हरणे कृते निवृत्ताहृतिस्तु जामितया तस्या
ऊर्ध्वगतास्यास्समसंख्या तद्दलम् गुणो/न्ते स्यात् तद्वर्गैरूपहतो हारो
व्यासाब्धिघातत: प्राग्वत् तस्याम् आप्तम् स्वमृणे कृते धाने शोधनान्च
करणीयम् सूक्ष्म: परिधि: सा स्यात् बहुकृत्वो हरणतो/तिसूक्ष्मास्च

yatsankhyaaatra harane krte nivrtta hrtis tu jamitaya
tasya urdhvagatasyas samasankhya taddalam guno ante syat
tadvargai rupahato haaro vyasabdhighatata: pragvat
tasyam aptam svamrne krte dhane sodhanan cha karaniyam
sukhma: paridhi: sa syat bahukrtvo haranato atisukshmas cha

.... Let the process stop at a certain stage, giving rise to a finite sum, multiply four times the diameter by half the even integer subsequent to the last odd integer used as divisor and then divide by the square of the integer increased by unity. The result is the correction to be added to or subtracted from finite sum. The choice of addition or subtraction is depending on sign of the last term in the sum. The final result is the circumference determined more accurately than by taking a large number of terms:

Circumference = 4D (1-1/3+1/5-1/7+.....)

Or $\Pi/4 = 1-1/3 + 1/5 - 1/7 + ... (n+1)½/(n+1)^2 + 1$

This is the contribution of Madhavacharya, even though now known in the name of a Western scholar.

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K. V. Sarma[73] in his book gives a theorem of Madhava, on higher level, for the approximation of the Π. It is known as De Moivres theorem. It is given in the Kriyakramakari on Lilavati by Madhava (Kanda 2, vrutta 40.

अस्मात् सूक्ष्मतरोऽन्यो विलिख्यते कश्चनापि संस्कार:
अन्ते सम संख्यादलवर्गसैको गुण: स एव पुन: ।
युगगुणितो रूपयुत: समसंख्यादलहतो भवेद् हार:
त्रिशरादि विषमसंख्या हरणात् परं एतत् एव वा कार्यम् ।।

Asmat sukshmataroanyo vilikhyate kashcanapi samskara:
ante samasankhyadalavarga saiko guna:, sa eva puna:
yugagunito rupayuta: samasankhyadalahato bhaved haara:
trisaradivisamashankhyaharanat param etad eve va karyam

A correction still more precise is being stated here. The multiplier is the square of half the even integer increased by unity. This multiplier multiplied by 4, then increased by unity and then multiplied by half the even integer is the divisor. This correction may be applied after the division by odd integers, 3, 5, etc. i.e

$$\text{Circumference} = 4D\,(1 - 1/3 + 1/5 - 1/7 \ldots\ldots + \ldots - 1/n(\tfrac{1}{2}(n+1)^2+1 \div ((\tfrac{1}{2}(n+1)^2 \times 4 + 1)\,(\tfrac{1}{2}(n+1))$$

Different from the Leibnitz series, Madhava has also developed another series for the approximation of Π which is given in Yuktibhasha as follows:

व्यासवर्गाद् रविहतात् पदम् स्यात् प्रथमं फलम्
तदादितास्त्रिसंख्याप्तम् फलम् स्याद् उत्तरोत्तरम् ।
रूपाद्ययुग्मसंख्याभिर् हृतेष्वेषु यथाक्रमम् विशमानम्
युतेस्त्यक्त्वा समम् हि परिधिर् भवेत् ।।

vyasavargad ravihatat padam syat prathamam phalam
tadaditas trisankhyaptam phalam syad uttarottaram
rupaddyayugmasankhyabhir hrteshu eshu yathakramam
visamanam yutestyaktva samam hi paridhir bhavet

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Multiply the square of the diameter by 12 and extract the square root of the product. That is the first term. Divide the first term by 3 to obtain the second, the second by 3 to obtain the third and so on and thus get further terms. Divide the terms, in order by the odd number 1,3,5, ... Add the odd order terms to and subtract the even order terms from the preceding. The result will give the circumference.

Circumference (Π D) =

$\sqrt{12D^2} - \sqrt{12D^2}/3 \times 3 + \sqrt{12D^2}/5 \times 3^2 - \sqrt{12D^2}/7 \times 3^2 + ...$

$\Pi = \sqrt{12}\ (1 - 1/3 \times 3 + 1/5 \times 3 \times 3 - 1/7 \times 5 \times 3 \times 3 ...)$

This can be further simplified by taking the common factor $\sqrt{12} \times D$, and the value of circumference or Π can be derived from that.

In Kriyakramakari, value of Π series is given by Madhavacharya (Kanda 2:40) as:

विबुधनेत्रगजाहि हुताशना त्रिगुण वेदभावारणा बाहव:
नव निखर्वमिते वृति विस्तरे परिधिमानम् इदम् जगदुर्बुधा:

vibudha-netra-gaj-ahi-hutasana tri-guna-veda-bha-varana-bahava:
nava-nikharva-mite vrtivistare paridhi-manam idam jagadur budda:

When for a diameter of 9^{11} units, the circumference is equal to 28 27433388233 units. From this the value of Π can be calculated as 3.14159265359, which is the same on the modern value of 3.14159265.

Tycho Brahe reduction of ecliptic of Achyutha Pisharati : In the mathematical calculations the reduction of the ecliptic has been reported to be derived by Tycho Brahe. However during the same period Achyutha Pisharati, a famous Kerala astronomer has also devloped the same reduction of ecliptic in his book Sputanirnaya.

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पातोनस्य विधोस्तु कोटिभुजयोर्जीवे मितस्ताडयेत्
अन्त्यक्षेपशराहतम् वधममुं विक्षेपकोट्याहरेत् ।
लब्धम् व्यासदलोघृतम् हिमकरे स्वर्णम् विपाते विधौ
युग्मायुग्मपदोपगे विधुरयम्स्पष्टो भगोले भवेत् ॥

Patonasya vidhostu kotibhujayorjive mithastadayet
antyakshepasarahatam vadhamamum vikshepakotyaharet
labdham vyasadaloddhrtam himakare svarnam, vipate vidhau
yugmayugmapadopage; vidhurayam spashto bhagole bhavet

Multiply the tabular cosine and sine of the moon minus node and the product by the tabular versine of the maximum latitude of the moon. Divide this by the tabular cosine of the latitude at the particular moment and the quotient is to be divided again by the tabular radius. The result is to be added to or subtracted from the moon's longitude, as the moon minus node is in an even or an odd quadrant, respectively. The true moon measured on the ecliptic is thus obtained.

Sarma has given the modern form of Achyuta's formula. If F is the longitudinal difference between the node and the planet, w the maximum latitude and y the actual latitude then the correction k = sin F. Cos F (1 - cos F)/cos y.

It is said that in Uparagakriayakrama, Achyuta has given another simplified formula for the reduction of ecliptic.

Infinite G.P. convergent of Nilakanta Newton's series : This series was originally contributed by Nilakanta Somayaji in his bhashaya. (Aryabhateeyabhashya Ganita 17)

एवं यस्तुल्यच्छेद परमभाग परमपर्यया अनन्ताया अपि संयोगः
तस्य अनन्तानाम् अपि कल्प्यमानस्य योगस्य आद्यावयविनः
परस्परमच्छेदाद् एकोनच्छेदामंश साध्यम् सर्वत्रापि समानम् एव .....

Evam yasthuthya ccheda paramabhaaga paramaparyayaa

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